Science & Technology Development Journal – Engineering and Technology, 3(SI3):141-149
Open Access Full Text Article

Research Article

Characterizations of variational convergence of bifunctions defined on products of two sets

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ABSTRACT
We present definitions of types of variational convergence of finite-valued bifunctions defined on rectangular domains and establish characterizations of these convergences. In the introduction, we present the origins of the research on variational convergence and then we lead to the specific problem of this paper. The content of the paper consists of 3 parts: variational convergance of fucntion; variational convergance of bifunction; and characterizations of variational convergence of bifunction, this part is the main results of this paper. In section 2, we presented the definition of epi convergence and presented a basic property problem that will be used to extend and develop the next two sections. In section 3, we start to present a new definition, the definition of convergence epi / hypo, minsup and maxinf. To clearly understand of these new definitions we have provided comments (remarks) and some examples which reader can check these definitions. The above contents serve the main result of this paper will apply in part 4. Now, we will explain more detail for this part as follows. Firstly, variational convergence of bifunctions is characterized by the epi- and hypo-convergence of related unifunctions, which are slices sup- and inf-projections.
The second characterization expresses the equivalence of variational convergence of bifunctions and the same convergence of the so-called proper bifunctions defined on the whole product spaces. In the third one, the geometric reformulation, we establish explicitly the interval of all the limits by computing formulae of the left- and right-end limit bifunctions, and this is necessary and sufficient conditions of the sequence bifunctions to attain epi / hypo, minsup and maxinf convergence.
Key words: Variation convergence, Epi-convergence, Hypo-convergence, Epi/hypo-convergence, Lopsided convergence, Equivalence class

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History
- Received: 9-12-2019
- Accepted: 20-11-2020
- Published: 31-12-2020

DOI : 10.32508/stdjet.v3iSI3.643

INTRODUCTION

Epi-convergence, epi/hypo-convergence, and lopsided convergence are the main types of variational convergence, of unifunctions (briefly, functions) and bifunctions, respectively. Epi-convergence was introduced independently [1–3]. For details of this convergence and its applications, the reader is referred to the books [4–6].

Epi/hypo-convergence of extended-real-valued bifunctions was proposed and developed [7,8]. Typical references for this convergence and its applications [9–12]. After the appearance [7], lopsided convergence, a modified verson stronger than epi/hypo-convergence was further studied [13]. All the aforementioned contributions investigate the class of extended-real-valued functions defined on the whole space X, denoted by fcn (X), or of bifunctions defined on the whole product space $X \times Y$ denoted by biv $(X \times Y)$. Lopsided convergence was discussed for the first time for a class of finite-valued bifunctions defined on $A \times B$, where $A \subset X$ and $B \subset Y$ with X, Y being finite-dimensional spaces [7].

This class is also called finite-valued bifunctions on rectangles and designated by fv-biv $(X \times Y)$. It is important because typical bifunctions in many mathematical models of practical problems are Lagrange functions [14,15] in constrained optimization, Hamilton functions in variational calculus and optimal control, and payoff functions in zero-sum games, all belong to this class. Epi/hypo-convergence of finite-valued bifunctions on rectangles was studied [16,17]. Lopsided convergence for special subclasses of fv-biv $(X \times Y)$ was investigated [18–20].

Lopsided convergence is stronger than epi/hypo-convergence. Hence, it has more beautiful properties. But, it is not symmetric with respect to the two components $x \in X$ and $y \in Y$. It has "one-sided" properties and is appropriated for either minsup- or maxinf-properties. However, the first variational properties of bifunctions attracting attention from researchers must be saddle-point properties [21–23], which combine both the minsup- and maxinf-properties. Especially, such properties are crucial for studies of dualities in general and dual problems [24] in particular. Observe

Cite this article : Diem H T H. **Characterizations of variational convergence of bifunctions defined on products of two sets**. *Sci. Tech. Dev. J. – Engineering and Technology;* 3(SI3):141-149.

further that recently lopsided convergence of finite-valued bifunctions and its applications were intensively studied for special types of bifunctions with applications mainly in special important classes of problems. However, for epi/hypo-convergence of finite-valued bifunctions, we see only papers [16,17] for the finite-dimensional case.

Motivated by the above brief about results on variational convergence obtained up to this time, in this paper we study variational convergence of finite-valued bifunctions on rectangles in the case of general metric spaces. Our aim is characterizations of such a convergence. The obtained results are new in various aspects or improve/extend the corresponding ones [16–18,25,26] for the finite-dimensional case to the general metric-space case.

Namely, we develop three characterizations of the epi/hypo- and lop-convergence of bifunctions in fv-biv in the general metric-space case. It should be stressed that characterizations of variational convergence are important not only for theoretical developments but in fact even more crucial for applications.

In this paper, all the spaces are metric spaces, if not otherwise specified. Our notation is standard. For instance, $\mathbb{R}$, $\bar{\mathbb{R}}$ and $\mathbb{N}$ stand for the real line, the extended real line $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, and the set of the natural numbers, respectively (resp). For $A \subset X$, int A and bd A denote the interior and the boundary of A, resp. Function $\varphi : X \to \bar{\mathbb{R}}$ is called lower semincontinuous (lsc, upper semincontinuous (usc)) at $\bar{x}$ if $liminf_{x\to\bar{x}}\varphi(x) \geq \varphi(\bar{x})$ ($limsup_{x\to\bar{x}}\varphi(x) \leq \varphi(\bar{x})$, resp). For $A^k \subset X$, the lower/inner limit and the upper/outer limit are defined by

$Liminf_k A^k := \left\{x \in X : lim_k x^k = x,\ for\ x^k \in A^k\right\}$,

$Limsup_k A^k := \left\{x \in X : lim_l x^{k_l} = x,\ for\ x^{k_l} \in A^{k_l}\right\}$,

x^{k_l} being subsequence of $x^k\}$.

If $\mathrm{Liminf}_k A^k = \mathrm{Limsup}_k A^k$, we say that A^k tend to A in the Painlevé-Kuratowski sense, denoted by $A^k \xrightarrow{P-K} A$ or $A = Lim_k A^k$. Later on, we also write simply li, ls, Li, Ls instead of liminf, limsup, Liminf, Limsup, resp.

VARIATIONAL CONVERGENCE OF FUNCTIONS

In this section, we present epi-convergence of functions and its basic variational properties. Let X be a metric space, A^k, $A \subset X$ be nonempty sets, and $\left\{\varphi^k : A^k \to \mathbb{R}\right\}_k$, $\varphi : A \to \mathbb{R}$ be elements of fv-fcn (X). The following definition extends the corresponding notion [26] from the case of finite-dimensional spaces to that of general metric spaces.

Definition 1 (epi-convergence) $\left\{\varphi^k\right\}_k$ in fv-fcn (X) is said to epi-converge to φ fv-fcn (X), denoted by $\varphi = e - lim_k \varphi^k$ or $\varphi^k \xrightarrow{e} \varphi$, if

(a) for all $x^k \in A^k \to x$, $\mathrm{li}_k \varphi^k(\mathrm{x}^k) \geq \varphi(\mathrm{x})$ when $x \in A$ and $\varphi^k(x^k) \to +\infty$ when $x \notin A$;

(b) for all $x \in A$, there exist $x^k \in A^k \to x$ such that $\mathrm{ls}_k \varphi^k(x^k) \leq \varphi(x)$.

Definition 2 (lower and upper epi-limits) Let $\varphi^k \in$ fv-fcn (X) and $x \in X$. The lower epi-limit of $\left\{\varphi^k\right\}_k$ at x is $\mathrm{eli}\varphi^k(x) := \inf_{\{x^k \in A^k \to x\}} \mathrm{li}_k \varphi^k(x^k)$, where $\left\{x^k \in A^k \to x\right\}$ means that we consider all the possible sequences $x^k \in A^k$ tending to x.

The upper epi-limit of $\left\{\varphi^k\right\}_k$ at x is $\mathrm{eli}\varphi^k(x) := \inf_{\{x^k \in A^k \to x\}} \mathrm{li}_k \varphi^k(x^k)$,

It can be proved that $\varphi^k \xrightarrow{e} \varphi$ if and only if $\mathrm{eli}\varphi^k(x) = \mathrm{els}\varphi^k(x) = \varphi(x)$ when $x \in A$ and $eli\varphi^k(x) = +\infty$ when $x \notin A$.

The notion symmetric to epi-convergence is hypo-convergence which is defined as follows. φ^k are called hypo convergent to φ, designated by $\varphi = h - lim_k \varphi^k$ or $\varphi^k \xrightarrow{h} \varphi$, if $-\varphi^k$ epi-converge to $-\varphi$. Hence, the lower and upper hypo-limits of $\left\{\varphi^k\right\}_k$ at $x \in A$ are, resp,

$\mathrm{hli}\varphi^k(x) := \sup_{\{x^k \in A^k \to x\}} \mathrm{li}_k \varphi^k\left(x^k\right)$,

$\mathrm{hls}\varphi^k(x) := \sup_{\{x^k \in A^k \to x\}} ls_k \varphi^k\left(x^k\right)$.

Similar to epi-convergence, $\varphi^k \xrightarrow{h} \varphi$ if and only if $\mathrm{hli}\varphi^k(x) = \mathrm{hls}\varphi^k(x) = \varphi(x)$ when $x \in A$ and $\mathrm{hli}\varphi^k(x) = -\infty$ when $x \notin A$.

Now we extend to the metric-space case the basic variational property of epi-convergence, proved [14] for the finite-dimensional case.

Theorem 1 (basic variational property of epi-convergence [6]) For $\varphi^k, \varphi \in$ fv − fcn (X) with $\varphi = \mathrm{e} - \lim \varphi^k$, one has $ls_k(\inf_{A^k} \varphi^k(x)) \leq \inf_A \varphi(x)$. Moreover, if x^{k_j} is a minimizer of φ^{k_j} for all $j \in N$ and $x^{k_j} \to \bar{x} \in A$, then $\bar{x}$ in a minimizer of φ and the infimal value of φ^{k_j} also converge to the infimal value of φ.

The proof is similar to the known proof for the finite-dimensional case and so omitted. The second part can be reformulated equivalently as follows: if $\mathrm{e} - \lim_k \varphi^k = \varphi$, then

$A \cap \mathrm{Ls_k}(\mathrm{argmin}_{A^k} \varphi^k) \subset \mathrm{argmin}_A(\varphi)$.

one can prove easily an extension of this inclusion: if $\varepsilon^k \searrow 0$, then

$A \cap \mathrm{Ls_k}(\varepsilon^k - \mathrm{argmin}_{A^k} \varphi^k) \subset \mathrm{argmin}_A(\varphi)$.

To have the reverse inclusion with the full Lim replacing Limsup together with the convergence of the corresponding infimal values, one needs the following tightness condition.

Definition 3 (tight epi-convergence) one says that $\left\{\varphi^k\right\}_k$ epi-converge tightly to φ in fv-fcn (X) if they epi-converge and for each $\varepsilon > 0$, there exists a compact set B_ε in X and $k_\varepsilon \in \mathbb{N}$ such that, for all $k \geq k_\varepsilon$, $\inf_{B_\varepsilon \cap A^k} \varphi^k \leq \inf_{A^k} \varphi^k + \varepsilon$.

Theorem 2 (basic property of tight epi-convergence[6])
(i) If φ^k tightly epi-converge to φ and $\inf_{A^k} \varphi$ is finite, then $\inf_{A^k} \varphi^k \rightarrow \inf_A \varphi$. The converge holds if X is finite-dimensional.
(ii) Assume that φ^k epi-converge to φ and $\inf_A \varphi$ is finite. If there exist $\varepsilon^k \searrow 0$ such that $A \cap Lim_k \varepsilon^k - \mathrm{argmin}_{A^k} \varphi^k = \mathrm{argmin}_A \varphi$, then φ^k tightly epi-converge to φ. The converse is true if X is separable.

To prove this statement, modify suitably the arguments for the finite-dimentional case[26] and apply the theorem on level sets associated with epi-convergence[27]. Due to the requirement of a limited length of the paper, we skip the details.

VARIATIONAL CONVERGENCE OF BIFUNCTIONS

For A, $A^k \subset X$, $B, B^k \subset Y$, $\phi^k : A^k \times B^k \rightarrow \bar{\mathbb{R}}$, and $\phi : A \times B \rightarrow$
R $, we propose the following definition.$

Definition 4 (epi/hypo-convergence) Bifunctions ϕ^k in fv-biv $(X \times Y)$ are called epi/hypo-convergent (e/h-convergent) to $\phi \in$ fv-biv $(X \times Y)$ if
(a) for all $y \in B$ and $x^k \in A^k \rightarrow x$, there exist $y^k \in B^k \rightarrow y$ such that $li_k \phi^k \left(x^k, y^k\right) \geq \phi(x,y)$ when $x \in A$ and $\phi^k \left(x^k, y^k\right) \rightarrow +\infty$ when $x \notin A$;
(b) for all $x \in A$ and $y^k \in B^k \rightarrow y$, there exist $x^k \in A^k \rightarrow x$ such that $ls_k \phi^k \left(x^k, y^k\right) \leq \phi(x,y)$ if $y \in B$ and $\phi^k \left(x^k, y^k\right) \rightarrow -\infty$ when $y \notin B$;
This convergence is denoted by $\phi = e/h - \lim_k \phi^k$ or $\phi^k \xrightarrow{e/h} \phi$.

Note that if φ^k do not depend on y, then epi/hypo-convergence becomes epi-convergence, and if they do not depend on x, it collapses to hypo-convergence. However, the epi/hypo-convergence of $\phi(\cdot,\cdot)$ is not the epi-convergence of $\phi(x,\cdot)$ to $\phi(\cdot,y)$ for all y together with the hypo-convergence of $\phi^k(x,\cdot)$ to $\phi(x,\cdot)$ for all x. This is a sufficient condition for e/h-convergence, but not a necessary condition. We see that e/h-convergence and h/e-convergence are symmetric. Moreover, h/e-convergence coincides with e/h-convergence, if one keeps taking minimum on x and maximum on y, only changes the order of the two operations. The lopsided convergence defined below is our extension to the metric-space case of the corresponding concept for the finite-dimensional case[26].

Definition 5 (minsup-lop convergence)
$\phi^k \in \mathrm{fv-biv}\ (X \times Y)$ are called minsup-lopsided (minsup-lop) convergent to $\phi \in \mathrm{fv-biv}\ (X \times Y)$ if
(a) for each $y \in B$ and $x^k \in A^k \rightarrow x$, there exist $y^k \in B^k \rightarrow y$ such that $li_k \phi^k \left(x^k, y^k\right) \geq \phi(x,y)$ when $x \in A$ and $\phi^k \left(x^k, y^k\right) \rightarrow +\infty$ when $x \notin A$;
(b) for all $x \in A$, there exist $x^k \in A^k \rightarrow x$ such that, for all $y^k \in B^k \rightarrow y$, $ls_k \phi^k \left(x^k, y^k\right) \leq \phi(x,y)$ when $y \in B$ and $\phi^k \left(x^k, y^k\right) \rightarrow -\infty$ when $y \notin B$.
This convergence is denoted by $\phi = \mathrm{mins} - \lim_k \phi^k$ or $\phi^k \xrightarrow{mins} \phi$.

Clearly the roles of x and y are not symmetric in this definition. Changing the order of x and y leads to the following definition of maxinf-lop convergence:
(a) for all $x \in A$ and $y^k \in B^k \rightarrow y$, there exist $x^k \in A^k \rightarrow x$ such that $ls_k \phi^k \left(x^k, y^k\right) \leq \phi(x,y)$ if $y \in B$ and $\phi^k \left(x^k, y^k\right) \rightarrow -\infty$ if $y \notin B$;
(b) for all $y \in B$ and $x^k \in A^k \rightarrow x$, there exist $y^k \in B^k \rightarrow y$ such that, for all $x^k \in A^k \rightarrow x$, $li_k \phi^k \left(x^k, y^k\right) \geq \phi(x,y)$ if $x \in A$ and $\phi^k \left(x^k, y^k\right) \rightarrow +\infty$ if $x \notin A$.
We denote this convergence by $\phi = \mathrm{maxi} - \lim_k \phi^k$, or $\phi^k \xrightarrow{maxi} \phi$.

It is obvious that each minsup-lop or maxinf-lop convergence implies e/h-convergence. Condition (a) of e/h-convergence and minsup-lop convergence are the same. While condition (b) of lop-convergence is stronger (b) of e/h-convergence. Indeed, for $x \in A$ lop-convergence requires the existence of a common sequence $x^k \in A^k \rightarrow x$ such that for all $y^k \in B^k \rightarrow y$, $ls_k \phi^k \left(x^k, y^k\right) \leq \phi(x,y)$ if $y \in B$ and $\phi^k \left(x^k, y^k\right) \rightarrow -\infty$ if $y \notin B$. Surely, this $\left\{x^k\right\}_k$ satisfies (b) of e/h-convergence. However, conversely, if one has $\phi^k \xrightarrow{e/h} \phi$, one still cannot derive that ϕ^k minsup-lop converge to ϕ as shown by the following example.

Example 1 Let $A^k = B^k = [1/k, 1], A = B = [0,1]$, and

$$\phi^k(x,y) = \begin{cases} 1\ if\ (x,y) \in A^k \times B^k, x \neq y, \\ 0\ if\ (x,y) \in A^k \times B^k, x = y. \end{cases}$$

Then,

$$\phi(x,y) = \begin{cases} 1\ if\ (x,y) \in [0,1]^2, x \neq y, \\ 0\ if\ (x,y) \in [0,1]^2, x = y. \end{cases}$$

Clearly $\phi^k \xrightarrow{e/h} \phi$. We show that condition (b) of minsup-lop convergence is violated. For $x = 0$ and any $x^k \in A^k \rightarrow x$, we take $y = 0$ and $y^k \in B^k \rightarrow 0$ such that $y^k \neq x^k$ for all k. Then, $ls_k \phi^k \left(x^k, y^k\right) = 1 > 0 = \phi(x,y)$.

Remark 1 (i) $\phi^k : A^k \times B^k \rightarrow \mathbb{R}$ are called convergent continously to $\phi : A \times B \rightarrow \mathbb{R}$ associated with $A^k \times B^k \xrightarrow{P-K} A \times B$ if for all $\left(x^k, y^k\right) \in A^k \times B^k \rightarrow (x,y)$, $\phi^k \left(x^k, y^k\right) \rightarrow \phi(x,y)$. This convergence is very strong. It implies all the three e/h-, minsup-lop, and maxinf-lop convergences. (Furthermore, the continuous convergence of ϕ implies also both the epi-convergnece and hypo-convergence of ϕ^k.) Hence, continuous convergence is a type of variational convergence. But, it is difficult to be satisfied.

(ii) Similar to the paper [16] for the finite-dimensionl case, we show in Theorem 5 that in our general case e/h-limits are not unique, but form (epi/hypo) equivalence classes. Fortunately, all the bifunctions in such a class have the same variational properties.

Note that the above definitions of variational convergence of bifunctions do not require that $A^k \xrightarrow{P-K} B^k \to A\times B$. That is why not only points $(x,y)\in A\times B$ are under consideration, but also all the points (x,y), which are limits of $\left(x^k, y^k\right)\in A^k\times B^k$, are taken into account.

In order to see the relation between fv-biv $(X\times Y)$ and biv $(X\times Y)$, given ϕ in fv-biv $(X\times Y)$, we define the corresponding two so-called proper bifunctions in biv $(X\times Y)$ as follows.

$$(\eta_{h/e}\phi)(x,y) := \begin{cases} \phi(x,y) \text{ if } (x,y)\in A\times B, \\ +\infty \quad \text{ if } y\in B,\ x\notin A, \\ -\infty \quad if\ y\notin B. \end{cases}$$

$$(\eta_{h/e}\phi)(x,y) := \begin{cases} \phi(x,y) \text{ if } (x,y)\,\widehat{\mathrm{I}}\,A\times B, \\ -\infty \quad \text{ if } x\in A,\ y\notin B, \\ +\infty \quad if\ x\notin A. \end{cases}$$

Then, we have in fact two “projections” $\eta_{e/h}$ and $\eta_{e/h}$ from fv-biv $(X\times Y)$ to biv $(X\times Y)$ transforming ϕ into the two corresponding proper bifunctions belonging to biv $(X\times Y)$. Looking at the definition of e/h-convergence, we see that beside $(x,y)\in A\times B$, we consider also points (x,y) with either $x\in A$ and $y\notin B$ or $x\notin A$ and $y\in B$. We call the points $(x,y)\in A\times B$ and these points the related points (through e/h-convergence). For all the related points, we have $\eta_{e/h}(x,y)=\eta_{h/e}(x,y)$. Because only the related points come into play, we will use the abbreviation η for both $\eta_{e/h}$ and $\eta_{h/e}$.

We also need to modify the definition of e/h-convergence of bifunctions of biv $(X\times Y)$ to have the corresponding Definition 6 below. For $\phi\in$ biv $(X\times Y)$, recall that its domain is defined by $\mathrm{dom}\phi = \mathrm{dom_x}\phi'\,\mathrm{dom_y}\phi$,

where $\mathrm{dom}_x\phi := \{x\in X|\phi(x,y)<+\infty, \forall y\in Y\}$, $\mathrm{dom}_y\Phi := \{y\in Y|\Phi(x,y)<-\infty, \forall x\in X\}$ Hence, ϕ is finite on $\mathrm{dom}\phi$, but ϕ also take finite values at different some points outside $\mathrm{dom}\phi$.

Definition 6 (epi/hypo-convergence, biv) Let ϕ and ϕ^k be in biv $(X\times Y)$. We say that ϕ^k e/h-converge to ϕ if

(a’) $\forall y \in dom_y\phi,\ \forall x^k \to x\in X, \exists y^k \to y,\ li_k\phi^k(x^k,y^k) \geq \phi(x,y);$

(b’) $\forall x \in dom_x\phi,\ \forall y^k \to y\in Y, \exists x^k \to x,\ ls_k\phi^k(x^k,y^k) \geq \phi(x,y).$

Remark 2 The definition of e/h-convergence [8] requires that (a’) and (b’) are fulfilled for all $y\in Y$ and $x\in X$ instead of $y\in dom_y\phi$ and $x\in dom_x\phi$. The following example shows that $\left\{\phi^k\right\}_k$ e/h-converge in the sense of Definition 6, but not e/h-converge following [8]. Assume that $X=Y=\mathbb{R}$ and ϕ^k are defined by

$$\phi^k(x,y)=\phi(x,y)= \begin{cases} 0 \text{ if } (x,y)\in[1,1]^2, \\ -\infty \text{ if } x\in(0,1),\ y\in[0,1] \\ +\infty \text{ in the other cases.} \end{cases}$$

Then, $\mathrm{dom_x}\phi=(0,1)$, $\mathrm{dom_y}\phi=(0,1)$ and ϕ^k e/h-converge to ϕ following our definition. However, ϕ^k do not satisfy (a) with $\forall y\in\mathrm{dom_y}\phi$ replaced by $\forall y\in Y$. Indeed, for y = 1 and $x^k=k^{-1}\to 0$, There do not exist $y^k\to -1$ with $li_k\phi^k(k^{-1},y^k)\geq\phi(0,-1)=+\infty$.

CHARACTERIZATIONS OF VARIATIONAL CONVERGENCE OF BIFUNCTIONS

Characterizations by epi- and hypo-convergence of related unifunctions

Proposition 1 (characterizations of e/h-convergence by slices)

(i) Condition (a) of e/h-convergence $\Leftrightarrow \forall x^k\in A^k\to x$, $\mathrm{hli}\phi^k(x^k,\cdot)\geq\phi(x,\cdot)$ with $\phi^k(x^k,\cdot)$ defined on A^k and $\phi(x,\cdot)$ on A, and $\forall x^k\in A^k\to x\notin A$, $\exists y^k\in B^k$, $\phi^k(x^k,y^k)\to+\infty \Leftrightarrow \forall x^k\in A^k\to x$, one has (b) of $\phi^k(x^k,\cdot)\xrightarrow{h}\phi(x,\cdot)$ with the domains defined as above, and $\forall x^k\in A^k\to x\notin A$, $\exists y^k\in B^k$, $\phi^k(x^k,y^k)\to+\infty$.

(ii) Condition (b) of e/h-convergence $\Leftrightarrow \forall y^k\in B^k\to y$, $\mathrm{els}\phi^k(\cdot,y^k)\geq\phi(\cdot,y)$ with $\phi^k(\cdot,y^k)$ defined on A^k and $\phi(\cdot,y)$ on A, and $\forall y^k\in B^k\to y\notin B$, $\exists x^k\in A^k$, $\phi^k(x^k,y^k)\to-\infty \Leftrightarrow \forall y^k\in B^k\to y$, one has (a) of $\phi^k(\cdot,y^k)\xrightarrow{e}\phi(\cdot,y)$ with the domains defined as above, and $\forall y^k\in B^k\to y\notin B$, $\exists x^k\in A^k$, $\phi^k(x^k,y^k)\to-\infty$.

Proof (i) We show only the first equivalence. Condition (a) of e/h-convergence is that for all $y\in B$ and $x^k\in A^k\to x$, there exist $y^k\in B^k\to y$ such that $li_k\phi^k(x^k,y^k)\geq\phi(x,y)$ when $x\in A$ together with the upper hypo-limit defined in part II. Moreover, the infinity condition in (a) of e/h-convergence is just the rest in the second side of this first equivalence. Similarly, applying this condition (a) together with the definition of hypo-convergence, we obtain the second equivalence.

(ii) The arguments are similar to those in (i) due to the symmetric property of epi/hypo-convergence. □

Proposition 2 (characterizations of minsup-lop convergence by slices)

(i) Condition (a) of misup-lop convergence is the same (a) of epi/hypo-convergence and so we have the same two equivalences in Proposition 1

(ii) Condition (b) of minsup-lop convergence $\Leftrightarrow \forall x \in A, \exists x^k \in A^k \to x$, such that $hls\phi^k(x^k,\cdot) \leq \phi(x^k,\cdot)$ with $\phi(x^k,\cdot)$ defined on B and $\min_{\{y^k \to y\}} \lim_k \phi^k\left(x^k,y^k\right) = -\infty$ when $y \in B$

Proof We need to prove only (ii). Condition (b) of minsup-lop convergence is $\forall x \in A$, $\exists x^k \in A^k \to x$, $\forall y^k \in B^k \to y$, $ls_k\phi^k\left(x^k,y^k\right) \leq \phi\left(x,y\right)$ if $y \in B$ and $\phi^k\left(x^k,y^k\right) \to -\infty$ if $y \notin B$ Combining this with the definition of lower hypo-limits, we complete the proof. □

Corollary 1: Let $\phi^k \xrightarrow{mins} \phi$.

(i) There exist $x^k \in A^k \to x \in A$ such that $\phi(x^k,\cdot) \xrightarrow{h} \phi(x,\cdot)$ on B and $\max_{\{y^k \in B^k \to y \in B\}} \lim_k \phi\left(x^k,y^k\right) = +\infty$ for all $x^k \in A^k \to x \notin A$;

(ii) If ϕ^k do not depend on y (x, resp), then $\phi\left(x^k,\cdot\right) \xrightarrow{e} \phi\left(x,\cdot\right)$ $\left(\phi\left(\cdot,y^k\right) \xrightarrow{h} \phi\left(\cdot,y\right)\right.$, resp).

Remark 3 (i) The characterization of maxinf-lop convergence in terms of slices clearly states as follows: for $\phi^k \xrightarrow{maxi} \phi$,

(a) there exist $y^k \in B^k \to y \in B$ such that $\phi\left(\cdot,y^k\right) \xrightarrow{e} \phi\left(\cdot,y\right)$ on A and min $\min_{\{x^k \in A^k \to x \in A\}} \lim_k \phi\left(x^k,y^k\right) = -\infty$ for all $y^k \in B^k \to y \notin B$;

(b) when ϕ^k do not depend on x (on y), $\phi\left(\cdot,\cdot\right) \xrightarrow{h} \phi\left(\cdot,\cdot\right)$ $\left(\phi\left(x^k,\cdot\right) \xrightarrow{e} \phi\left(x,\cdot\right)\right.$, resp)., resp).

(ii) We also have an assertion corresponding to (i) in Corollary 1, but with attention on changes on the roles of (x,y) and of epi- and hypo-convergence. However, there is no direct connection between the lop-convergence of ϕ^k to ϕ and $\phi\left(\cdot,y\right) \xrightarrow{e} \phi\left(\cdot,y\right)$ for all y and $\phi(x,\cdot) \xrightarrow{h} \phi(x,\cdot)$ for all x. If ϕ^k depend only on one component, (ii) in Corollary 1 becomes a sufficient condition for the lop-convergence to ϕ.

Next, we discuss a relation between lop-convergence of bifunctions and convergence of other unifunctions being its sup- and inf-projections defined as follows. The sup-projection (inf-projection, resp) of $\phi \in$ fv-biv $(X \times Y)$ is $\zeta : A \to \mathbb{R} \cup \{+\infty\}$ ($\eta : B \to \mathbb{R}$, resp) defined by, $\zeta(x) := \sup_{y \in B} \phi(x,y)$ ($\eta(y) := \inf_{x \in A} \phi\left(x,y\right)$, resp).

To establish this relation, we need the following concept of ancillary tight convergence.

Definition 7 (ancillary tight e/h- and lop-convergence)

(i) (x-ancillary tight e/h-convergence) ϕ^k are called x-ancillary e/h-convergent to ϕ if (a) of e/h-convergence and the following condition are satisfied

(be) $\forall y \in B$, $\forall x \in A$, $\forall y^k \in B^k \to y, \exists x^k \in A^k \to x$, $\mathrm{ls}_k\zeta^k(x^k) \leq \zeta(x)$.

(ii) (y-ancillary tight e/h-convergence) ϕ^k are called y-ancillary e/h-convergent to ϕ if (b) of e/h-convergence is fulfilled together with the condition

(ae) $\forall x \in A, y \in B, \forall x^k \in A^k$, $\exists y^k \in B^k \to y, \mathrm{li}_k\eta^k(y^k) \geq \eta(y)$.

(iii) (-ancillary tight minsup-lop convergence) ϕ^k are called x-ancillary tightly minsup-lop convergent to ϕ if one has (a) of minsup-lop convergence and (b) strenghthened as

(b1) condition (b) holds and, for any positive ε and any $x^k \in A^k \to x$ given in (b), there exist a compact set $B_e \subset Y$ and k_ε such that $\sup_{B_\varepsilon \subset B^k} \phi^k(x^k,...) \geq \sup_{B^k} \phi^k(x^k,...) - \varepsilon$, $\forall k \geq k_\varepsilon$.

(iv) (y-ancillary tight maxninf-lop convergence) ϕ^k are said to y-ancillary tight maxinf-lop converge to ϕ if if one has (a) of maxinf-lop convergence and (b) strenghthened as

(b2) Condition (b) holds and, for any positive ε and any $y^k \in B^k \to y$ given in (b), there exists a compact set $A_\varepsilon \subset X$ and k_ε such that$\inf_{A_\varepsilon \cap A^k} \phi^k(...,y) \leq \sup_{A^k} \phi^k(...,y^k) + \varepsilon$, $\forall k \geq k_\varepsilon$.

Proposition 3 (x-ancillary tight e/h-convergence and epi-convergence of sup-projections) Let the e/h-convergence of ϕ^k to ϕ in fv-biv $(X \times Y)$ be x-ancillary tight and ζ^k, ζ be sup-projections of ϕ^k, ϕ, resp, such that dom$\zeta \neq 0$ and dom$\zeta^k \neq 0$. Then, $\mathrm{e} - \lim_k \zeta^k = \zeta$.

Proof To check condition (a) of epi-convergence for ζ^k, let $x^k \in$ dom$\zeta^k \to x$. First, assume that $x^k \in$ domζ^k. For each $\varepsilon > 0$, choose $y_\varepsilon \in B$ such that $\phi\left(x,y_\varepsilon\right) \geq \xi\left(x\right) - \varepsilon$. By (a) in Definition 4, there are $y_\varepsilon^k \in B^k \to y_\varepsilon$ such that

$\mathrm{li}_k\zeta^k(x^k) \geq \mathrm{li}_k\phi(x^k,y^k) \geq \phi(x,y_\varepsilon) \geq \zeta\left(x\right) - \varepsilon$. As ε is arbitrary, $\mathrm{li}_k\zeta^k(x^k) \geq \zeta(x)$..

If $\zeta(x) = +\infty$, for each $\gamma > 0$, choose $y_\gamma \in B$ such that $\phi(x,y_\gamma) \geq \gamma$. Again by the above (a), there are $y_\gamma^k \in B^k \to y_\gamma$ with $\mathrm{li}_k\zeta^k(x^k) \geq \mathrm{li}_k\phi(x^k,y_\gamma^k) \geq \gamma$. By the arbitrariness of γ, $\zeta^k(x^k) \to +\infty$.

To verify condition (b) of the epi-convergence, for any x^k Îdomζ take arbitrarily $y^k \in B^k \to y \in B$. By condition (be), there exist $x^k \in A^k \to x$ such that $\mathrm{ls}_k\zeta^k(x^k) \leq \zeta(x)$. Hence, for some k_0 and all $k \geq k_0$, $\zeta^k(x^k)$ is finite, i.e., $x^k \in$ domζ^k and so the aforementioned (b) is fulfilled. □

Proposition 4 (x-ancillary tight minsup-lop convergence and epi-convergence of sup-projections) Assume that ϕ^k minsup-lop converge x-ancillary tightly to ϕ, dom$\zeta \neq 0$ and dom$\zeta^k \neq 0$. Then, ζ^k epi-converge to ζ.

Proof We prove the equivalent conclusion: epi$\zeta^k \xrightarrow{P-K}$ epiζ. To see $\mathrm{Ls}_k(\text{epi}\zeta^k) \subset$ epiζ, take

any (x,α) in the left-hand side, which means there are k_j such that $\left(x^{k_j},\alpha^{k_j}\right) \in \mathrm{epi}\zeta^{k_j}$ tend to (x,α). We claim that $x \in \mathrm{dom}\zeta$. Clearly, $x \in A$, because otherwise condition (a) of minsup-lop convergence yields $y^{k_j} \in B^{k_j}$ such that $\phi^{k_j}(x^{k_j},y^{k_j}) \to +\infty$, which contradicts the fact that $\alpha^{k_j} \geq sup_{B^{k_j}}\phi^{k_j}(x^{k_j},\cdot)$ and $\alpha^{k_j} \to \alpha$. Suppose $\zeta(x) = +\infty$. Then, for any $\gamma > \alpha$, there exists $y_\gamma \in B$ with $\phi\left(x,y_\gamma\right) > \gamma$. In view of the above (a), there exist $y_\gamma^{k_j} \in B^{k_j} \to y_\gamma$ such that $\mathrm{li}_j\phi^{k_j}(x^{k_j},y^{k_j}) \geq \phi(x,y_\gamma)$. Hence,
$\alpha = \lim_j \alpha^{k_j} \geq \mathrm{li}_j\zeta^{k_j}(x^{k_j}) \geq \mathrm{li}_j\phi^{k_j}(x^{k_j},y^{k_j}) \geq \phi(x,y_\gamma) > \gamma > \alpha$. This contradiction shows that $x \in \mathrm{dom}\zeta$.
For any $\varepsilon > 0$, there exists $y_\varepsilon \in B$ with $\phi(x,y_\varepsilon) \geq \zeta(x) - \varepsilon$. The aforementioned (a) again gives $y_\varepsilon^{k_j} \in B^{k_j} \to y_\varepsilon$ such that
$\mathrm{li}_j\zeta^{k_j}(x^{k_j}) \geq \mathrm{li}_j\phi^{k_j}(x^{k_j},y^{k_j}) \geq \phi(x,y_\varepsilon) \geq \zeta(x) - \varepsilon$.
Therefore, as ε is arbitrary,
$\alpha = \lim_j \alpha^{k_j} = \mathrm{li}\zeta^{k_j}(x^{k_j}) \geq \zeta(x)$.
So, $\mathrm{Ls}_k(\mathrm{epi}\zeta^k) \subset \mathrm{epi}\zeta$. To prove the inclusion $\mathrm{epi}\zeta \subset \mathrm{Li}_k\mathrm{epi}\zeta^k$, observe that ϕ^k minsup-lop convergent x-ancillary tightly to ϕ implies that for any $x \in A$, there exists $x^k \in A^k$ ensuring the tight hypo-convergence of $\phi^k(x^k,\cdot)$ to $\phi(x,\cdot)$. Using the counterpart of Theorem 2 (i) for hypo-convergence, one has $\zeta^k(x^k) \to \zeta(x)$. For any $(x,\alpha) \in \mathrm{epi}\zeta$, i.e., $\alpha \geq \zeta(x)$, clearly,
$(x^k,\zeta^k(x^k) + \alpha - \zeta(x)) \in \mathrm{epi}\zeta^k \to (x,\alpha)$. Therefore, $\mathrm{epi}\zeta \subset \mathrm{Li}_k\mathrm{epi}\zeta^k$. □
We omit the statements about inf-projections corresponding to the above propositions.

Characterizations by proper bifunctions

Theorem 3 For ϕ,ϕ^k in fv-biv $(X\times Y)$, ϕ^k e/h-converge to ϕ if and only if $\eta\phi^k$ e/h-converge to $\eta\phi$.
Proof We use the explicit formula of $\eta_{e/h}$, but denotes it by η for simplicity.
(a') of Definition 6 ⇒ (a) of Definition 4. Suppose $y \in B = \mathrm{dom}_y(\eta\phi)$ and $x^k \in A^k \to x$. Then, (a') yields $x^k \in A^k \to x$ such that

$$\mathrm{li}_k(\eta\phi^k)(x^k,y^k) \geq (\eta\phi)(x,y). \tag{1}$$

Consider first $x \in A$. Suppose the existence of $y^{k_j} \notin B^{k_j}$. Then, $(\eta\phi^{k_j})(x^{k_j},y^{k_j}) = -\infty$, contradicting (1). Hence, $y^k \in B^k$ and (1) means that $\mathrm{li}_k\phi^k(x^k,y^k) \geq \phi(x,y)$ as (a) requires. Now let $x \notin A$. Then, $(\eta\phi)(x,y) = +\infty$ and (1) implies that $(\eta\phi^k)(x^k,y^k) \to +\infty$. Again $y^{k_j} \notin B^{k_j}$ would give $(\eta\phi^{k_j})(x^{k_j},y^{k_j}) \equiv -\infty$, contradicting (1). Hence, $\phi^k(x^k,y^k) \equiv (\eta\phi^k)(x^k,y^k) \to +\infty$ as (a) requires.
(a) ⇒ (a'). Let $y \in B = \mathrm{dom}_y(\eta\phi)$ and $x^k \to x \in X$. If $x \in A$ and there exist $x^{k_j} \in A^{k_j}$, then (a) give $y^{k_j} \in B^{k_j} \to y$ such that $li_j(\eta\phi^{k_j})(x^{k_j},y^{k_j}) = li_j(\eta\phi^{k_j})(x^{k_j},y^{k_j}) \geq (\eta\phi)(x,y)$. (2)

We build a sequence y^k by inserting, for k with $x^k \notin A^k$, $y^k \in B^k$ so that $y^k \to y$. Then, for k with $x^k \notin A^k$, $(\eta\phi^k)(x^k,y^k) \to +\infty$ and does not effect the liminf in (2). Hence, we have (1). In the case that $x^k \notin A^k$, we can take any $y^k \in B^k \to y$ to obtain $(\eta\phi^k)(x^k,y^k) \to +\infty$ and then get (1).
If $x^k \notin A^k$, as $y \in B$ one has $(\eta\phi)(x,y) \to +\infty$. If, for the given $\left\{x^k\right\}_k$, $x^k \notin A^k$, we take arbitrarily $y^k \in B^k \to y$ to have $(\eta\phi^k)(x^k,y^k) \to +\infty$ and get (1). Otherwise, there is a subsequence $x^{k_j} \in A^{k_j}$. Then, (a) yields $y^{k_j} \in B^{k_j} \to y$ such that $\phi^{k_j}(x^{k_j},y^{k_j}) \to +\infty$. For the additional k with $x^k \notin A^k$, we add to $\left\{y^{k_j}\right\}$ points $y^k \in B^k$ so that $y^k \to y$. Because, for additional k, $(\eta\phi^k)(x^k,y^k) \to +\infty$, we have the entire convergent sequence $\phi^k(x^k,y^k) \to +\infty$ and so (1).
(b) ⇒ (b') Let $x \in A$ and $y^k \in B^k \to y$. (b') supplies $x^k \to x$ such that

$$\mathrm{ls}_k(\eta\phi^k)(x^k,y^k) \leq (\eta\phi)(x,y) \tag{3}$$

If $y \in B_k$ and there is a subsequence $x^{k_j} \notin A^{k_j}$, then $\phi^{k_j}(x^{k_j},y^{k_j}) \equiv +\infty$ contradicting (3). Thus, $x^k \in A^k$ as (b) requires. If $x^k \in A^k$, then (3) implies that $\mathrm{ls}_k(\eta\phi^k)(x^k,y^k) \leq \phi(x,y)$ as (b) requires. If $y \notin B$, $(\eta\phi)(x,y) = +\infty$. Hence, we get a contradiction if there is a subsequence $x^{k_j} \notin A^j$. Therefore, $x^k \in A^k$ and we obtain (b).
(b) ⇒ (b '). This is similar to (a) ⇒ (a') combined with (b') ⇒ (b). □

Geometric characterizations

To characterize e/h- and lop-convergence of $\left\{\phi^k\right\}_k$ to ϕ in a geometric way, we introduce the following six limit bifunctions of $\left\{\phi^k\right\}_k$ in connection with ϕ.
Definition 8 (six limit bifunctions) Let ϕ and ϕ^k be in fv-biv$(X\times Y)$ and $(x,y) \in X\times Y$.
(i) (lower lop-limit bifunctions)
$\mathrm{lli}\phi^k(x,y) := \sup_{\{y^k\in B^k\to y\}}\inf_{\{x^k\in A^k\to x\}}\mathrm{li}_k\phi^k(x^k,y^k)$
for $y \in B$,
$\mathrm{lli}\phi^k(x,y) := \sup_{\{y^k\in B^k\to y\}}\inf_{\{x^k\in A^k\to x\}}\mathrm{ls}_k\phi^k(x^k,y^k)$
if $y \notin B$
(ii) (upper lop-limit bifunctions)
$\mathrm{lls}\phi^k(x,y) := \inf_{\{x^k\in A^k\to x\}}\sup_{\{y^k\in B^k\to y\}}\mathrm{ls}_k\phi^k(x^k,y^k)$
for $x \in A$,
$\mathrm{lls}\phi^k(x,y) := \inf_{\{x^k\in A^k\to x\}}\sup_{\{y^k\in B^k\to y\}}\mathrm{li}_k\phi^k(x^k,y^k)$
if $x \notin A$.
(iii) (lower and upper e/h-limit bifunctions)
$\mathrm{ehli}\phi^k(x,y) := \sup_{\{y^k\in B^k\to y\}}\inf_{\{x^k\in A^k\to x\}}\mathrm{ls}_k\phi^k(x^k,y^k)$
if $y \in B, x \in A$,
$\mathrm{ehls}\phi^k(x,y) := \inf_{\{x^k\in A^k\to x\}}\sup_{\{y^k\in B^k\to y\}}\mathrm{li}_k\phi^k(x^k,y^k)$
if $x \in A, y \in B$.
Remark 4 Straightforwardly from the definitions one always has

$\text{ehls}\phi^k(x,y) \le \text{lls}\phi^k(x,y)$ for $x \in A,\ y \in B$
$\text{lli}\phi^k(x,y) \le \text{ehli}\phi^k(x,y)$ for $x \in A,\ y \in B$
In Theorem 6 below, we will see that, just due to these inequalities, both minsup-lop and maxinf-lop limits are unique if exist. Moreover, one also has
$\text{lli}\phi^k(x,y) \le \text{lls}\phi^k(x,y)$ for $(x,y) \in A \times B$
$\text{ehls}\phi^k(x,y) \le \text{lls}\phi^k(x,y)$ for $(x,y) \in A \times B$
$\text{lli}\phi^k(x,y) \le \text{ehls}\phi^k(x,y)$ for $(x,y) \in A \times B$

Theorem 4 (formulas of the six limit bifunctions) Let ϕ and ϕ^k be in fv-biv $(X\,x\,Y)$ and $(x,y) \in X \times Y$.
(i) (lower lop-limit bifunctions)
$\text{lli}\phi^k(x,y) := \max_{\{y^k \in B^k \to y\}} \inf_{\{x^k \in A^k \to x\}} \text{li}_k \phi^k(x^k,y^k)$ for $y \in B$,
$\text{lli}\phi^k(x,y) := \sup_{\{y^k \in B^k \to y\}} \min_{\{x^k \in A^k \to x\}} \text{ls}_k \phi^k(x^k,y^k)$ if $y \notin B$,
(ii) (upper lop-limit bifunctions)
$\text{lls}\phi^k(x,y) := \min_{\{x^k \in A^k \to x\}} \sup_{\{y^k \in B^k \to y\}} \text{ls}_k \phi^k(x^k,y^k)$ for $x \in A$,
$\text{lls}\phi^k(x,y) := \inf_{\{x^k \in A^k \to x\}} \max_{\{y^k \in B^k \to y\}} \text{li}_k \phi^k(x^k,y^k)$ if $x \notin A$.
(iii) (lower and upper e/h-limit bifunctions)
$\text{ehli}\phi^k(x,y) := \sup_{\{y^k \in B^k \to y\}} \min_{\{x^k \in A^k \to x\}} \text{ls}_k \phi^k(x^k,y^k)$ if $y \in B, x \in A$,
$\text{ehls}\phi^k(x,y) = \inf_{\{x^k \in A^k \to x\}} \max_{\{y^k \in B^k \to y\}} \text{li}_k \phi^k(x^k,y^k)$ if $x \in A, y \in B$.

Proof For reasons of similarity, we discuss only (ii). To prove the first formula, denote $u := \text{l}ls\phi^k(x,y)$. If $u = +\infty$, any $\{x^k \in A^k \to x\}$ is a minimizer for the considered expression. If $-\infty < u < +\infty$, for any $\varepsilon^j \searrow 0$, there exist $x_j^k \in A^k \to x$ such that for all $y_j^k \in B^k \to y$, $\text{ls}_k \phi^k(x_j^k,y_j^k) \le u + \varepsilon^j$.. We take ls_j to get $\text{ls}_j \text{ls}_k \phi^k(x_j^k,y_j^k) \le u$. Due to Corollary A3 in [7], for each k there is *j(k)* such that $\text{ls}_k \phi^k(x_{j(k)}^k, y_{j(k)}^k) \le u$. Hence, the sequence $\left\{x_{j(k)}^k\right\}_k$ is a minimizer. If $u = -\infty$, for any $\theta^j \to +\infty$, one finds $x_j^k \in A^k \to x$ such that for all $y_j^k \in B^k \to y$, $\text{ls}_k \phi^k(x_j^k,y_j^k) \le -\theta^j$. Similarly as above, with that corollary we see that the obtained $\left\{x_{j(k)}^k\right\}_k$ is a minimizer.

Consider the second formula. Fix a sequence $\{x^k \in A^k\}_k \to A$ and denote
$s := \sup_{\left\{y^k \in B^{k_j}\right\}} \text{li}_k \phi^k(x^k,y^k)$
We check the achievement of the involved supremum. If $s = -\infty$, any $\{y^k \in B^k\}_k$ gives the maximum. If $-\infty < s < +\infty$, for any $\varepsilon^j \searrow 0$, there exist $y^k \in B^k$ with $\text{li}_k \phi^k(x^k,y_j^k) \ge s - \varepsilon^j$ for each $j \in \mathbb{N}$. Taking li_j we have $\text{li}_j \text{li}_k \phi^k(x^k,y_j^k) \ge s$. In view of Lemma A.1[8], we have *j(k)* for each k such that $\text{li}_k \phi^k(x^k, y_{j(k)}^k) \ge s$. Of course the sequence $\left\{y_{j(k)}^k\right\}_k$ gives the aforementioned maximum. By the arbitrariness of $\{x^k\}_k$, we obtain the required equality. If $s = +\infty$, then for any $\theta^j \to +\infty$, one finds $y_j^k \in B^k$ such that $\text{li}_k \phi^k(x^k,y_j^k) \ge \theta^j$ for any j. Taking li_j, one arrives at $\text{li}_j \text{li}_k \phi^k(x^k,y_j^k) = +\infty$. Again by Lemma A.1[8], one gets $j(k)$ for each k such that $\text{li}_k \phi^k(x^k, y_{j(k)}^k) = +\infty$. Hence, $\left\{y_{j(k)}^k\right\}_k$ is a maximizer for the sup-expression in the definition of s. □

Theorem 5 (characterization of e/h-convergence) Let ϕ and ϕ^k be in fv-biv $(X \times Y)$. Then, ϕ^k e/h-converge to ϕ if and only if the following two assertions hold
(a) $\phi(x,y) \le \text{ehls}\phi^k(x,y)$ when $x \in A$ and $y \in B$, and $\text{lls}\phi^k(x) = +\infty$ when $x \notin A$;
(b) $\text{ehli}\phi^k(x,y) \le \phi(x,y)$ when $y \in B$ and $x \in A$, and $\text{lli}\phi^k(x) = +\infty$ when $y \notin B$.

Proof From Definition 4 and this theorem, it is obvious that (a) and (b) of e/h-convergence coincide with (a) and (b), resp, of Theorem 5. □

Observe that for ϕ^k fv-biv $(X \times Y)$, which of $\text{ehli}\phi^k(x,y)$ and $\text{ehls}\phi^k(x,y)$ is smaller changes from point to point in general. Theorem 5 stipulates that if $\text{ehli}\phi^k(x,y) \le \text{ehls}\phi^k(x,y)$ for all $(x,y) \in A \times B$, any ϕ satisfying the inequality $\text{ehli}\phi^k(x,y) \le \phi \le \text{ehls}\phi^k(x,y)$ on $A \times B$ and fulfills also the infinity items (for x outside A or y outside B) is an e/h-limit. Thus, in general the limits of a given e/h-convergent sequence, if exist, form an e/h-equivalence class of bifunctions.

Theorem 6 (characterizations of lop-convergence) Let ϕ and ϕ^k be in fv-biv $(X \times Y)$.
(i) ϕ^k minsup-lop converge to ϕ if and only if: for $x \in A$, $\text{lls}\phi^k(x,y) \le \phi(x,y) \le \text{ehls}\phi^k(x,y)$ if $y \in B$, and $\text{lls}\phi^k(x,y) = -\infty$ if $y \notin B$, and for $x \notin A$, $\text{lls}\phi^k(x) = +\infty$. Hence, we have in fact the equalities for $(x,y) \in A \times B$ by Remark 4.
(ii) ϕ^k maxinf-lop converge to ϕ if and only if: for $y \in B$, $\text{ehli}\phi^k(x,y) \le \phi(x,y) \le \text{lli}\phi^k(x,y)$ if $x \in A$, and $\text{lli}\phi^k(x,y) = +\infty$ if $x \notin A$, and for $y \notin B$, $\text{lli}\phi^k(y) = -\infty$. Hence, we have in fact the equalities for $(y,x) \in A \times B$ by Remark 4.

Proof (i) Following the second formula in Theorem 4 (ii) and (iii), condition (a) of minsup-lop convergence means that $\phi(x,y) \le \text{ehls}\phi^k(x,y)$ if $x \in A, y \in B$, and $\text{lli}\phi^k(x) = +\infty$ if $x \notin A$, which is a part of assertion (i). For $x \in A$ and $y\,\hat{\text{I}}\,Y$, take any $\varepsilon^j \searrow 0$ and $\gamma^j \nearrow +\infty$. We define

$$\theta^j(x,y) := \begin{cases} \phi(x,y) + \varepsilon^j & \text{if } y \in B, \\ -\gamma^j & \text{if } y \notin B. \end{cases}$$

The remaining part of (i) means that, for $x \in A$ and $y \in Y$,

$$\text{lls}\phi^k(x,y) \le \begin{cases} \phi(x,y) & \text{if } y \in B, \\ -\infty & \text{if } y \notin B. \end{cases}$$

This is equivalent to the statement: for each j, there exist $x_j^k \in A^k \to x$, for all $y_j^k \in B^k \to y$, $\mathrm{ls}_k \phi^k(x_j^k, y_j^k) \leq \theta^j(x,y)$. Taking ls_j of both sides yields

$$\mathrm{ls}_j \mathrm{ls}_k \phi^k(x_j^k, y_j^k) \leq \begin{cases} \phi(x,y) \text{ if } y \in B, \\ -\infty \quad \text{if } y \notin B. \end{cases}$$

By virtue of Corollary A3 in [8], one has $j(k)$ for each k such that

$$\mathrm{ls}_k \phi^k(x_{j(k)}^k, y_{j(k)}^k) \leq \begin{cases} \phi(x,y) \text{ if } y \in B, \\ -\infty \quad \text{if } y \notin B. \end{cases}$$

Therefore, the sequence $\{x^k\}_k := \left\{x_{j(k)}^k\right\}_k$ is a minimizer for the expression of $\mathrm{lls}\phi^k(x,y)$, i.e., we obtain the remaining part of assertion (i).

(ii) The proof is similar to (i), but applying Theorem 4 with the first formula of (iii), both the formulas of (i), and Lemma A1 [8] instead of the formulas of Theorem 4 applied in (i) and Corollary A37 , resp. □

By Theorem 6, we see that, if exists, each of the minsup-lop limit and maxinf-lop limit is unique. Furthermore, both minsup-lop and maxinf-lop limits exist at the same time if and only if, for $(x,y) \in A \times B$, $\mathrm{lli}\phi^k = \mathrm{ehli}\phi^k = \phi = \mathrm{ehls}\phi^k = \mathrm{lls}\phi^k$ together with the four infinity conditions stated in Theorem 6.

ACKNOWLEDGMENTS

This research was supported by the Project of Vietnam national University of type C under grant number C2019-20-05. I am deeply indebted to the anonymous reviewer for the valuable remarks and suggestions, which helped me so much in completing this final version.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interest in publishing this paper.

AUTHORS' CONTRIBUTIONS

This study is finished by one author.

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