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*Research Article*

# Necessary optimality conditions in nonsmooth semi-infinite multiobjective optimization under metric subregularity

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**ABSTRACT**

We consider nonsmooth semi-infinite multiobjective optimization problems under mixed constraints, including infinitely many mixed constraints by using Clarke subdifferential. Semi-infinite programming (SIP) is the minimization of many scalar objective functions subject to a possibly infinite system of inequality or/and equality constraints. SIPs have been proved to be very important in optimization and applications. Semi-infinite programming problems arise in various fields of engineering such as control systems design, decision-making under competition, and multi-objective optimization. There is extensive literature on standard semi-infinite programming problems. The investigation of optimality conditions for these problems is always one of the most attractive topics and has been studied extensively in the literature. In our work, we study optimality conditions for weak efficiency of a multi-objective semi-infinite optimization problem under mixed constraints including infinitely many of both equality and inequality constraints in terms of Clarke sub-differential. Our conditions are the form of the Karush-Kuhn-Tucker (KKT) multiplier. To the best of our knowledge, only a few papers are dealing with optimality conditions for SIPs subject to mixed constraints. By the Pshenichnyi-Levin-Valadire (PLV) property and the directional metric sub-regularity, we introduce a type of Mangasarian-Fromovitz constraint qualification (MFCQ). Then we show that (MFCQ) is a sufficient condition to guarantee the extended Abadie constraint qualification (ACQ) to satisfy. In our constraint qualifications, all functions are nonsmooth and the number of constraints is not necessarily finite. In our paper, we do not need the involved functions: convexity and differentiability. Later, we apply the extended Abadie constraint qualification to get the KKT multipliers for weak efficient solutions of SIP. Many examples are provided to illustrate some advantages of our results. The paper is organized as follows. In Section Preliminaries, we present our basic definitions of nonsmooth and convex analysis. Section Main Results prove necessary conditions for the weakly efficient solution in terms of the Karush-Kuhn-Tucker multiplier rule with the help of some constraint qualifications.

**Key words:** Optimality condition, SIP, constraint qualification, weak efficiency, metric subregularity

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## INTRODUCTION

Semi-infinite optimization (SIP) is the simultaneous minimization of finitely many scalar objective functions under an arbitrary set of inequality constraints or/and equality constraints. (SIPs) arise in many fields of applied mathematics such as robotics, control system design, etc, see for instance [1–3]. Investigation of optimality conditions for SIPs has been considered extensively in the literature.

With linear semi-infinite systems, Goberna [4] introduced he Farkas-Minkowski property, Puenten and Vera [5] proposed the local Farkas-Minkowski property and used it as a constraint qualification to get Lagrange multipliers. For convexsemi-infinite optimization, many constraint qualifications have been studied in Lopez and Vercher [6]. With the help of the Abadie constraint qualification, optimality conditions for semi-infinite systems of convex and linear inequalities were developed in Li [7]. For smooth problems, Stein [8] proposed the Abadie and Mangasarian-Fromovitz constraint qualifications to conisder optimality conditions. By employing variational analysis, Mordukhovich and Nghia [9] obtained necessary conditions under the extended perturbed Mangasarian-Fromovitz and Farkas–-Minkowski constraint qualification. For nonsmooth problem with inequality constraints, Zheng and Yang [10] employed the directional derivative to obtain Lagrange multiplier rules. Kanzi and Nobakhtian [11,12] introduced several nonsmooth analogues of the Abadie constraint qualification and the Pshenichnyi-Levin-Valadire property and applied them to obtain optimality conditions. Chuong [13] proposed the limiting constraint qualification in terms of the Mordukhovich subdifferential and applied it to optimality conditions. Kanzi [14] investigated nonsmooth semi-infinite problems with mixed constraints by the Michel-Penot subdifferential. We observe that

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the constraints in the above mentioned papers contain finitely many equalities. There are very few publications dealing with infinitely many equality constraints.

In this paper we investigate opyimality conditions for weak efficiency of a multiobjective semi-infinite optimization problem under mixed constraints including infinitely many of both equality and inequality constraints in terms of Clarke subdifferential. By the Pshenichnyi-Levin-Valadire (PLV) property and the directional metric subregularity, we propose Mangasarian-Fromovitz constraint qualification (MFCQ) to guarantee the extend Abadie constraint qualification (ACQ) to satisfy. In our constraint qualifications, all functions are nonsmooth and the number of the equality constraints is not necessary finite. Then, we apply them to get the KKT multipliers. The paper is organized as follows. In Section *Prelininaries*, we present our basic definitions. Section *Main results* prove necessary conditions for weak efficiency in terms of Karush-Kuhn-Tucker multiplier under some constraint qualification.

## RELIMINARIES

$\mathbb{N}, \mathbb{R}^n$ and $\mathbb{R}^n_+$ stand for the set of the natural numbers, a n-dimensional vector space and its nonnegative orthant, respectively (resp). $B(x,r)$ denotes the open ball with centre $x$ and radius $r$. For $M \subset \mathbb{R}^n$, int$M$, $clM$, $bdM$ and $coM$ stand for its interior, closure, boundary, and convex hull of $M$, resp. The cone hull of $M$ is defined by $coneM := \{\lambda x??? , x \in M\}$. The contingent cone of $M \subset \mathbb{R}^n$ at $\bar{x} \in clM$ is

$$T(M,\bar{x}) := \{d \in \mathbb{R}^n,\ t_n\, 0 \to 0^+, \exists d_n \to d, \bar{x}+t_n d_n \in M, \forall n \in \mathbb{N}\}.$$

A map $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $x_0 \in \mathbb{R}^n$ if there is a neighborhood U of $x_0$ and a real number $x_0 \in \mathbb{R}^n$ such that

$||f(x) - f(y)|| \le L||x-y||, \forall x, y \in U.$

A set-valued map $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ is concave if

$\forall a, b \in \mathbb{R}^n,\ \forall \lambda \in [0,1],$

$\lambda F(a) + (1-\lambda) F(b) \subseteq F(\lambda a + (1-\lambda) b).$

**Definition 2.1.** ([15])

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $x_0, d \in \mathbb{R}^n$. The Clarke directional derivative of $f$ at $x_0$ in direction $d$ is

$f^0(x_0, d) := \limsup_{x \to x_0, t \to 0^+} \frac{f(x+td)-f(x)}{t}$

and the Clarke subdifferential of $f$ at $x_0$ is

$\partial_C f(x_0) :=$
$\left\{x^* \in \mathbb{R}^n\ ??f^0(x_0,d) \ge \langle x^*, d\rangle, \forall d \in \mathbb{R}^n\right\}.$

Recall that the directional of $f$ at $x_0$ in direction $d$ is

$f'(x_0, d) := \limsup_{t \to 0^+} \frac{f(x_0+td)-f(x_0)}{t}$

$f$ is regular at $x_0$ if $f^0(x_0,d) = f'(x_0,d)$..

The following properties will be useful in the sequel ([15]).

**Proposition 2.1**

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x_0 \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$.

(i) $d \mapsto f^0(x_0, d)$ is finite, positivel homogeneous and subadditive on $\mathbb{R}^n$, and $\partial\left(f^0(x_0,\cdot)\right)(0) = \partial_C f(x_0)$, where $\partial$ denotes the subdifferential in the sense of convex analysis.

(ii) $\partial_C f(x_0)$ is a nonempty, convex and compact subset of $\mathbb{R}^n$ and, for every

$d \in \mathbb{R}^n, f^0(x_0,d) = max_{x^* \in \partial_C f(x_0)} \langle x^*, d\rangle.$

(iii) $\partial_C(f+g)(x_0) \subseteq \partial_C f(x_0) + \partial_C g(x_0)$. If in addition both $f$ and $g$ are regular at $x_0$, then the equality holds.

(iv) If $x_0$ is a local minimum of $f$, then $0 \in \partial_C f(x_0)$.

Besides single-valued directional derivatives, we need the following set-valued directional derivatives.

**Definition 2.2**

The Hadamard set-valued directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $x_0 \in \mathbb{R}^n$ in direction $d_0 \in \mathbb{R}^n$ is

$$Df(x_0,d) := \{y \in \mathbb{R}^m ??t_n \to 0^+, \exists d_n \to d, y = \lim_{n\to\infty} t_n^{-1}(f(x_0+t_n d_n) - f(x_0))\}$$

**Definition 2.3** ([16])

$f : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $y_0 = f(x_0)$ .$f$ is said to be-directionally metrically subregular at $x_0$ in direction $d$ if there are a neighborhood $U$ of $x_0$, $a \ge 0$, and $r > 0$, for $t \in (0,r)$ and $v \in B_X(d,r)$ ,$d\left(x_0+tv, f^{-1}(y_0)\right) \le ad(y_0, f(x_0+tv))$.

**Proposition 2.2**

$f : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$, and $y_0 = f(x_0)$. If $0 \notin Df(x_0)(d)$ then $f$ is directionally metrically subregular at $x_0$ in direction $d$.

***Proof.*** Suppose there are $t_n \to 0$ and $d_n \to d$ such that, for all $n$,

$d\left(x_0+t_n d_n, f^{-1}(y_0)\right) > nd(y_0, f(x_0+t_n d_n))$. Then, there exists $y_n = f(x_0+t_n d_n)$ such that

$||y_n - y_0|| < n^{-1}||(x_0+t_n d_n) - x_0||,$

$t_n^{-1}||y_n - y_0|| < n^{-1}||d_n||.$

By setting $v_n = t_n^{-1}(y_n - y_0)$,, one has $v_n \to 0$ and $y_0 + t_n v_n = f(x_0+t_n d_n)$, i.e., $0 \in Df(x_0)(d)$, which contradicts the assumption. □

The following example present that the sufficient condition given in Proposition 2.2 is not necessary.

**Example 2.1** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1,x_2) = |x_1 - x_2|$ and $d_1 = (1,0)$. We can check that $0 \notin Df(0)(d_1) = \{1\}$, hence the assumption of Proposition 2.2 is fulfilled. By calculations, we have that

$d\left(tv, f^{-1}(0)\right) = 2^{-1/2} t|v_1 - v_2|,$

$d(0, f(tv)) = t|v_1 - v_2|$

for $v=(v_1,v_2)\in\mathbb{R}^2$ and $t\in\mathbb{R}_+$ with $tv\notin f^{-1}(0)$. Then, for $r>0$, $t\in(0,r)$, and $v\in B(d_1,r)$, $\frac{t}{\sqrt{2}}|v_1-v_2|\le t|v_1-v_2|$,
*i.e.*, $d\left(tv,f^{-1}(0)\right)\le d(0,f(tv))$.
Hence, $f$ is directionally metrically subregular at 0 in direction $d_1$ in Proposition 2.2. Now we replace $d_1$ by $d_2=(1,1)$. Similarly, we check that the above inequality holds for $r>0, t\in(0,r)$, and $v\in B(d_2,r)$. However, $0\in Df(0,0)(d_2)=\{0\}$ ..

## MAIN RESULTS

We investigate the fmultiobjective semi-infinite optimization problem under mixed constraints:

$$\text{(P)} \min{}^{\mathbb{R}^m_+} \text{ s.t.} \begin{cases} g_i(x)\le 0,\ i\in I \\ h_j(x)\le 0,\ j\in J \end{cases}$$

where $f:=(f_1,\ ...,\ f_m):\mathbb{R}^n\to\mathbb{R}^m$, $g_i:\mathbb{R}^n\to\mathbb{R}$ for $i\in I$, and $h_j:\mathbb{R}^n\to\mathbb{R}$ for $j\in J$, are locally Lipschitz. The index sets $I$ and $J$ are arbitrary. The feasible set of problem (P) is

$$\Omega:=\{x\in\mathbb{R}^n ? \begin{Bmatrix} g_i(x)\le 0,\ i\in I \\ h_j(x)\le 0,\ j\in J \end{Bmatrix}.$$

**Definition 3.1** For the problem (P) and $x_0\in\Omega$. $x_0$ is called a local weak efficient solution of (P), written as $x_0\in LW(P)$, if there is a neighborhood $U$ of $x_0$ such that
$(f(U\cap\Omega)-f(x))\cap(-\text{int}\mathbb{R}^m_+)=\varnothing$.
We denote $I(x_0):=\{i\in I ? g_i(x_0)=0\}$ and

$$L(\Omega,x_0):= \left\{d\in X ? \begin{Bmatrix} g_i^0(x_0,d)\le 0,\forall i\in I(x_0) \\ h_i^0(x_0,d)=0,\forall j\in J(x_0) \end{Bmatrix}\right\},$$

$$\triangle:=\bigcup_{i\in I(x_0)}\partial_C g_i(x_0)\cup \bigcup_{j\in J}\left(\partial_C h_j(x_0)\cup\partial_C(-h_j)(x_0)\right),$$

$G(x):=sup_{i\in I}g_i(x)$, $H(x):=sup_{j\in J}max\{h_j(x),-h_j(x)\}$.

**Definition 3.2** ([5]) The Pshenichnyi-Levin-Valadire (PLV) property holds at $x_0\in\Omega$ with respect to (wrt) $G$ iff $G$ is locally Lipschitz around $x_0$ and $\partial_C G(x_0)\subset conv\bigcup_{i\in I(x_0)}\partial_C g_i(x)$.
If I is finite and $g_i$ are locally Lipschitz around $x_0$ for $i\in I$, obviously the problem (P) has the Pshenichnyi-Levin-Valadire (PLV) property at $x_0\in\Omega$ wrt $G$. Sufficient conditions for $G$ to be locally Lipschitz were considered[17].

**Definition 3.3** For (P) and $x_0\in\Omega$.
(i) The extended Abadie constraint qualification (ACQ) satisfies at $x_0$ if $L(\Omega,x_0)=T(\Omega,x_0)$.
(ii) The extend Mangasarian-Fromovitz constraint qualification (MFCQ) satisfies at $x_0$ if there exists $\bar{d}$ such that
(a) $g_i^0(x_0,\bar{d})<0$ for all $i\in I(x_0)$;
(b) $H$ is directionally metrically subregular at $x_0$, $DH(x_0,\cdot)$ is concave on $X$, $h_j$ is regular for all $j\in J$, and $0\in DH\left(x_0,\bar{d}\right)$.

**Theorem 3.1** *If (P) has the (PLV) property at $x_0\in\Omega$ wrt G and the (MFCQ) satisfies at $x_0$, then the (ACQ) satisfies at $x_0$.*

***Proof.*** By the (MFCQ), there is $\bar{d}$ such that $g_i^o(x_0,\bar{d})<0$ for all $i\in I(x_0)$. This implies that
$\langle x^*,\bar{d}\rangle<0,\forall x^*\in\bigcup_{i\in I(x_0)}\partial_C g_i(x)$,
$\langle x^*,\bar{d}\rangle<0,\forall x^*\in conv(\bigcup_{i\in I(x_0)}\partial_C g_i(x))$.
By (PLV), one has $\langle x^*,\bar{d}\rangle<0$ for all $x^*\in\partial_C G(x_0)$ and so $G_0\left(x_0,\bar{d}\right)<0$. Then, $\lim\sup_{t\to 0^+}\frac{G^0(x_0+t\bar{d})-G(x_0)}{t}\le G^0\left(x_0,\bar{d}\right)<0$, which implies there are $\beta$ and $\varepsilon$ such that
(1) $G\left(x_0+t\bar{d}\right)-G(x_0)<-t\beta,\forall t\in(0,\varepsilon)$. Besides, as $0\in DH\left(x_0,\bar{d}\right)$, there exist $t_n\to 0^+$, $d_n\to\bar{d}$, and $z_n\to 0$ such that $t_nz_n\in H(x_0+t_nd_n)$. The metric subregularity of $H$ gives $a\ge 0$ such that, for large $n$,

$$d\left(x_0+t_nd_n,H^{-1}(0)\right)\le ad(0,H(x_0+t_nu_n)) \le at_n||z_n||.$$

Hence, there exist $\bar{d}_n$ and $\varepsilon$ with $t_n^{-1}\varepsilon_n\to 0^+$ such that $x_0+t_n\bar{d}\in H^{-1}(0)$ and $||(x_0+t_nu_n)-(x_0+t_n\bar{u}_n)||\le at_n||z_n||+\varepsilon_n$.
Then, $\bar{d}_n\to d$. Since $x_0+t_n\bar{d}_n\in H^{-1}(0)$, one has,
$max\left\{h_j\left(x_0+t_n\bar{d}_n\right),-h_j\left(x_0+t_n\bar{d}_n\right)\right\}\le 0$,
Hence, for large $n$,
(2) $h_j(x_0+t_n\bar{d}_n)=0,\forall j\in J$.
From (1), one has , for large $n$,
$G\left(x_0+t_n\bar{d}_n\right)-G(x_0)<-t_n\beta$.
Since $G$ is locally Lipschitz at $x_0$, there is $L>0$ such that, for large $n$,
$G\left(x_0+t\bar{d}_n\right)-G\left(x_0+t_n\bar{d}_n\right)\le Lt_n\left|\left|\bar{d}_n-\bar{d}\right|\right|$,

$$G\left(x_0+t\bar{d}_n\right)\le G\left(x_0+t\bar{d}_n\right)+Lt_n\left|\left|\bar{d}_n-\bar{d}\right|\right| < G(x_0)+t_n\left(-\beta+L\left|\left|\bar{d}_n-\bar{d}\right|\right|\right)\le 0.$$

This implies that $g_i\left(x_0+t\bar{d}_n\right)\le 0$ for all $i\in I$. By combining this and (2), one has $x_0+t\bar{d}_n\in\Omega$. Hence, $\bar{d}\in T(\Omega,x_0)$.
Let $d\in L(\Omega,x_0)$, we prove $d\in T(\Omega,x_0)$.
Set $d_n=n^{-1}\bar{d}+\left(1-n^{-1}\right)d$ for $n\ge 2$.
By Proposition 2.1, for all $i\in I(x_0)$ one has
(3) $g_i^0(x_0,d_n)\le n^{-1}g_i^0\left(x_0,\bar{d}_n\right)+\left(1-n^{-1}\right)g_i^0\left(x_0,\bar{d}_n\right)<0$. Since $h_j$ is regular at $x_0$ and $d\in L(\Omega,x_0)$, one gets for all $j\in J$, $h_j^{'}(x_0,d)=h_j^0(x_0,d)=0$ and
$\lim_{t\to 0^+}\frac{h_j(x_0+td)-h_j(x_0)}{t}=\lim_{t\to 0^+}\frac{h_j(x_0+td)}{t}=0.$

Then, there exists $t_n \to 0^+$ such that $\lim_{n\to\infty} \frac{max\{h_j(x_0+t_nd),-h_j(x_0+t_nd)\}}{t_n} = 0$. Hence, $0 \in DH(x_0,d)$.

Because $DH(x_0,\cdot)$ is concave,

$$n^{-1}DH(x_0,\bar{d}) + \left(1-n^{-1}\right)DH(x_0,d) \in DH\left(x_0, n^{-1}\bar{d} + \left(1-n^{-1}\right)d\right).$$

Hence,

(4) $0 \in DH(x_0,d_n)$

From (3) and (4), similar to the above arguments, one has $d_n \in T(\Omega,x_0)$. As $d_n \to d$ and is a closed cone, $d \in T(\Omega,x_0)$.

The proof is complete. □

**Remark 3.1**

Nonsmooth SIPs involving mixed constraints[9,14], the (MFCQ) was used to consider a number of equality constraints. In these paper, the functions were continuously differentiable with the linearly independent gradients such that $\langle \nabla f_j(x_0), \bar{d}\rangle = 0$ for $j \in J$. The inequality constraints were continuously differentiable and the equalities werestrictly differentiable. By employing directional metric subregularity, out (MFCQ) can be used to nonsmooth infinite mixed constraint systems and the condition $0 \in DH\left(x_0,\bar{d}\right)$ can be applied in many cases..

The next example provides a case where Theorem 3.1 can be employed, while many Mangasarian-Fromovitz-type constraint qualifications cannot.

**Example 3.1** Let $g_i, h_j : \mathbb{R}^2 \to \mathbb{R}$ be defined by for $i \in N$

$$g_1(x_1,x_2) = x_1, g_{2i+1}(x_1,x_2) = x_1 - i^{-1},$$
$$g_2(x_1,x_2) = x_2, g_{2i+2}(x_1,x_2) = x_2 - (1+i)^{-1}, h_j(x_1,x_2) = j(x_1-x_2),\ j \in (0,1).$$

Hence, $\Omega = \left\{(x_1,x_2) \in \mathbb{R}^2 | x_1 = x_2 \le 0\right\}$.

For $x_0 = (0,0)$, $I(x_0) = \{1,2\}$. We see that $G(x_1,x_2) = sup\{x_1,x_2\}$, $H(x_1,x_2) = |x_1-x_2|$ are locally Lipschitz at $x_0$ and $\partial_C G(x_0) \subseteq conv \bigcup_{i\in I(x_0)} \partial_C g_i(x)$. Thus, (P) has the (PLV) property at $x_0$ wrt $G$. Now, we check that the (MFCQ) is fulfilled at $x_0$ with $\bar{d} = (1,-1)$. For $i \in I(x_0)$, $j \in J$, $g_i^0(x_0,\bar{d}) = -1 < 0$, $h_j$ is regular, $H(x_1,x_2) = |x_1-x_2|$ and so $H$ is directionally metrically subregular at $x_0$ (by Example 2.1), $DH(x_0,(d_1,d_2)) = conv\{(d_1,-d_2);(-d_1,d_2)\}$ for all $(d_1,d_2) \in X$ and so $DH(x_0,\cdot)$ is concave, and $0 \in DH(x_0,\bar{d})$. Therefore, the (MFCQ) holds at $x_0$. By Theorem 3.1, the (ACQ) holds at $x_0$. (We can check the (ACQ) by direct calculations as follows. As $g_1^0(x_0,d) = d_1$, $g_2^0(x_0,d) = d_2$, and $h_j^0(x_0,d) = j(d_1-d_2)$, we have

$$L(\Omega,x_0) = T(\Omega,x_0) = \left\{(d_1,d_2) \in \mathbb{R}^2 ? d_1 = d_2 \le 0\right\}$$

and so (ACQ) holds. Because J infinite, the (MFCQ)[9,14] cannot be employed.

The following example shows the essentialness of the directional metric subregularity of H.

**Example 3.2** Let $g_i$ be the same as in Example 3.1 and $h_j(x_1,x_2) = j\left(x_1^2 - x_2^2\right)$, $j \in (0,1)$.

Hence, $\Omega = \left\{(x_1,x_2) \in \mathbb{R}^2 ?? \tau_1 = \tau_2 \le 0\right\}$ and $I(x_0) = \{1,2\}$ for $x_0 = (0,0)$. Similar to Example 3.1, (P) has the (PLV) property at $x_0$ wrt $G$. We check that the (MFCQ) holds at $x_0$ for $\bar{d} = (1,-1)$. We have $g_i^0\left(x_0,\bar{d}\right) = -1 < 0$ for all $i \in I(x_0)$.

$h_j$ is regular, $j \in J$, $H(x_1,x_2) = \left|x_1^2 - x_2^2\right|$, $DH(x_0,(d_1,d_2)) = \{0\}$ for $(d_1,d_2) \in X$ and so $DH(x_0,\cdot)$ is concave, and $0 \in DH\left(x_0,\bar{d}\right)$. On the other hand, as $g_1^0(x_0,d) = 0$, $g_2^0(x_0,d) = d_2$, and $h_j^0(x_0,d) = 0$, we have

$$L(\Omega,x_0) = \left\{(d_1,d_2) \in \mathbb{R}^2 ? d_1 \le 0, d_2 \le 0\right\},$$
$$T(\Omega,x_0) = \left\{(d_1,d_2) \in \mathbb{R}^2 ? d_1 = 0, d_2 = 0\right\},$$
$$L(\Omega,x_0) \ne T(\Omega,x_0).$$

The cause is that $H$ is not directionally metrically subregular at $x_0$. We have

$d\left(tv, H^{-1}(0)\right) = 1/\sqrt{2}\min\{|v_1+v_2|, |v_1-v_2|\}$ and $d(0,H(tv)) = t^2\left|v_1^2 - v_2^2\right|$ for $v = (v_1,v_2) \in \mathbb{R}^2$ and $t \in \mathbb{R}_+$ with $tv \notin H^{-1}(0)$. Then, the subregularity means that for any $a,r > 0$, $t \in (0,r)$, and $v \in B_X\left(\bar{d},r\right)$,

$$\frac{t}{\sqrt{2}} min\{|v_1+v_2|, |v_1-v_2|\} \le at^2\left|v_1^2 - v_2^2\right|.$$

But, this does not hold.

Now, by employ the extend ACQ, we present a necessary optimality condition for weak efficiency of problem (P), as follows.

**Theorem 3.2** *Let $x_0$ be a local weak efficiency of (P). If the (ACQ) holds at $x_0$, $\triangle$ is closed, and $f_k$ is regular and Lipschitz around $x_0$, for $k = 1,\dots m$, then there exist $(\alpha_1,\dots,\alpha_m) \in \mathbb{R}_+^m \backslash \{0\}$, $\beta_i \ge 0$ for $i \in I(x_0)$, and $\gamma_j \ge 0$ for $j \in J$ such that*

$$0 \in \sum_{k=1}^m \alpha_k \partial_C f_k(x_0) + \sum_{i\in I(x_0)} \beta_i \partial_C g_i(x_0) + \sum_{j\in J} \gamma_j \left(\partial_C h_j(x_0) \cup \partial_C\left(-h_j\right)(x_0)\right).$$

**Proof. Step 1.** We claim that the system

$$\begin{cases} f_1^0(x_0,d) < 0, \dots, f_m^0 < 0 \\ d \in T(\Omega,x_0) \end{cases}$$

has no solution. Suppose that there is $d \in T(\Omega,x_0)$ satisfying $f_t^0(x_0,d) \le 0$ for all $t = 1,2,\dots,m$. By setting $y = \left(f_1^0(x_0,d),\dots,f_m^0(x_0,d)\right)$, one has $y \in -int\mathbb{R}_+^m$. As $d \in T(\Omega,x_0)$, there exist $t_n \to 0^+$ and $d_n \to d$ such that $x_0 + t_nd_n \in \Omega$ for all $n \in \mathbb{N}$. Since $f_k$ is regular and locally Lipschitz at $x_0$, one has

$$\lim_{n\to\infty} \frac{f_k(x_0+t_nd) - f_k(x_0)}{t_n} = f_k^0(x_0,d),$$
$$\lim_{n\to\infty} \frac{f_k(x_0+t_nd_n) - f_k(x_0+t_nd)}{t_n} = 0,$$
$$\lim_{n\to\infty} \frac{f_k(x_0+t_nd) - f_k(x_0)}{t_n} = \lim_{n\to\infty} \frac{f_k(x_0+t_nd) - f_k(x_0) + f_k(x_0+t_nd_n) - f_k(x_0+t_nd)}{t_n}.$$

Hence, $\lim_{n\to\infty} \frac{f(x_0+t_nd_n) - f(x_0)}{t_n} = y$.

As $y \in -\text{int}\mathbb{R}^m_+$, for large $n$, one has $f(x_0 + t_n d_n) - f(x_0) \in -\text{int}\mathbb{R}^m_+$, which is a contradiction. Therefore, the mentioned system has no solution.

**Step 2.** From Step 1, by Theorem 3.13 in [18], we have $(\alpha_1, .., \alpha_m) \in \mathbb{R}^m_+$, such that

$\sum_{k=1}^m \alpha_k f_k^0(x_0, d) \geq 0, \ \forall d \in T(\Omega, x_0).$

Because (ACQ) holds, one has

(5) $\sum_{k=1}^m \alpha_k f_k^0(x_0, d) \geq 0, \forall d \in L(\Omega, x_0).$

**Step 3.** Denote

$\mathrm{A} = cone(conv(\triangle))$

and the indicator function of $\mathrm{A}^0$ by $\delta_{\mathrm{A}^0}$.

Then, (5) implies that

$\sum_{k=1}^m \alpha_k f_k^0(x_0, d) \geq 0, \forall d \in \mathrm{A}^0.$

Because $0 \in \mathrm{A}^0$ and $\sum_{k=1}^m \alpha_k f_k^0(x_0, 0) = 0$,

$0 \in argmin_{d \in X}\left\{\sum_{k=1}^m \alpha_k f_k^0(x_0, d) + \delta_{\mathrm{A}^0}(d)\right\}.$

By Proposition 2.1, $f_k^0(x_0, \cdot)$ is continuous and convex and $\mathrm{A}^0$ is convex. Therefore ([19]),

$0 \in \partial\left(\sum_{k=1}^m \alpha_k f_k^0(x_0, \cdot) + \delta_{A^0}(\cdot)\right)(0).$

By the sum rule of subdifferentials, one has

(6) $0 \in \partial\left(\sum_{k=1}^m \alpha_k f_k^0(x_0, \cdot)\right)(0) + \partial\delta_{A^0}(\cdot)(0)$

Since $\partial f_k^0(x_0, \cdot)(0) = \partial_C f_k^0(x_0)$, one gets

$\partial\left(\sum_{k=1}^m \alpha_k f_k^0(x_0, \cdot)\right)(0) = \sum_{k=1}^m \alpha_k \partial_k f_k(x_0).$

As $\triangle$ is closed, by the bipolar theorem, one has $\partial\delta_{\mathrm{A}^0}(0) = \left(\mathrm{A}^0\right)^0 = \mathrm{A}$. Hence, from (6), there exist $(\alpha_1, ..., \alpha_m) \in \mathbb{R}^m_+ \setminus \{0\}$, $\beta_i \geq 0$ for $i \in I(x_0)$, and $\gamma_j \geq 0$ for $j \in J$ such that

$$0 \in \sum_{k=1}^m \alpha_k \partial_k f_k(x_0) + \sum_{i \in I(x_0)} \beta_i \partial_C g_i(x_0) + \sum_{j \in J} \gamma_j \left(\partial_C h_j(x_0) \cup \partial_C(-h_j)(x_0)\right).$$

The proof is complete. □

**Example 3.3**

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ with $f = (f_1, f_2)$ and $g_i, h_j: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_1(x_1, x_2) = \begin{cases} x_1 + x_2 \ if \ x_1 \geq 0, \\ x_1^2 + x_2 \ if \ x_1 < 0, \end{cases} \quad f_2(x_1, x_2) = x_2,$$

$$g_1(x_1, x_2) = x_2, g_i(x_1, x_2) = x_1^2 x_2 - \tfrac{1}{i}, i \in \mathbb{N} \setminus \{1\},$$

$$h_j(x_1, x_2) = j x_1^3 - x_1 x_2, \ j \in (0, 1).$$

Let $x_0 = (0, 0)$. $I(x_0) = \{1\}$. By direct computations, one has

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 ? x_1 = 0, x_2 \leq 0\right\},$$

$$T(\Omega, x_0) = \left\{(d_1, d_2) \in \mathbb{R}^2 ? \ d_1 = 0, d_2 \leq 0\right\},$$

$$\partial_C f_1(x_0) = \{(y, 1), \ y \in [0, 1]\}, \ \partial_C f_2(x_0) = \{(0, 1)\},$$

$$\partial_C g_1(x_0) = \{(0, 1)\}, \partial_C g_i(x_0) = (0, 0)\}, \ i \in \mathbb{N} \setminus \{1\},$$

$$\partial_C h_1(x_0) = \{(0, 0)\}, j \in (0, 1).$$

We can check that $f_1, f_2$ are regular at $x_0$ and $L(\Omega, x_0) = \left\{(d_1, d_2) \in \mathbb{R}^2 \ \ d_1 = 0, d_2 \leq 0\right\} = T(\Omega, x_0)$. Thus the (ACQ) holds. Now we apply Theorem 3.2. If $x_0$ is a local weak efficiency then there are $\alpha_1, \alpha_2 \in \mathbb{R}^2_+ \setminus \{(0, 0)\}$, $\beta_i > 0$ for $i \in I(x_0) = 1$, and $\gamma_j \geq 0$ for $j \in J$ such that

$$0 \in \sum_{l=1,2}^m \alpha_k \partial_k f_k(x_0) + \sum_{i \in I(x_0)} \beta_i \partial_C g_i(x_0) + \sum_{j \in J} \gamma_j \left(\partial_C h_j(x_0) \cup \partial_C(-h_j)(x_0)\right).$$

$$\alpha_1(y, 1) + \alpha_2(0, 1) + \beta_1(0, 1) = (0, 0)$$

Consequently $\alpha_1 + \alpha_2 + \beta_1 = 0$, a contradiction. According to Theorem 3.2, $(0, 0)$ is not a local weak efficiency of (P).

## LIST OF ABBREVIATION

ACQ: Abadie constraint qualification
KKT: Karush-Kuhn-Tucker
MFCQ: Mangasarian-Fromovitz constraint qualification
PLV: Pshenichnyi-Levin-Valadire
SIP: Semi-infinite multiobjective optimization

## CONFLICT OF INTEREST

We declare that there is no conflict of whatsoever involved in publishing this research.

## AUTHOR S' CONTRIBUTIONS

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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