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Research article

Optimality conditions and duality with new variants of generalized derivatives and convexity

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ABSTRACT

In this paper, a general vector optimization problem with inequality constraints is considered, this topic is very popular and important model with a long research history in optimization. The generality of setting is mainly expressed in the following three factors. The underlying spaces being linear spaces without topology (except the decision space being additionally equipped with this structure in some results). The "orderings" in both objective and constraint spaces are defined by arbitrary nonempty sets (not necessarily convex cones). The problem data are nonsmooth mappings, i.e., they are not Fréchet differentiable. For this problem, the optimality conditions and Wolfe and Mond-Weir duality properties are investigated , which lie at the heart of optimization theory. These results are established for the three main and typical optimal solutions: (Pareto) minimal, weak minimal, and strong minimal solutions in both local and global considerations. The research define a type of Gateaux variation to play the role of a derivative. For optimality conditions, and introduce the concepts of on-set differentiable quasiconvexity for global solutions and sequential differentiable quasiconvexity for local ones. Furthermore, each of them is separated into type 1 for sufficient optimality conditions and type 2 for necessary ones.

After obtaining optimality conditions, applying them to derive weak and strong duality relations for the above types of solutions following our duality schemes of the Wolfe and Mon-Weir types. Due to the complexity of the research subject: considerations of duality are different from that of optimality conditions, we have to design two more appropriate types of generalized quasiconvexity: scalar quasiconvexity for the weak solution and scalar strict convexity for the Pareto solution.

So all the results are in terms of the aforementioned Gateaux variation and various types of generalized quasiconvexity.

The results are remarkably different from the related known ones with some clear advantages in particular cases of applications.

Key words: general vector optimization problem, Gateaux variation, sequential differentiable quasiconvexity, on-set differentiable quasiconvexity, necessary optimality conditions, sufficient optimality conditions, scalar generalized quasiconvexity, scalar generalized strict convexity, Wolfe duality, Mond-Weir duality

INTRODUCTION

This paper is devoted to optimality conditions and duality relations for vector optimization which are two topics among those in the center of this area. There have been an enormous number of contributions to optimality conditions and duality in the literature, the first seminal work [1] and so optimality conditions usually appear as Kuhn-Tucker optimality conditions or rules (the name "Karush-Kuhn-Tucker" may be more frequently used recently after the earlier preliminary result of Karush becoming known). Note that Fritz John optimality conditions are another weaker type of results. Here, we discuss both types: Karush-Kuhn-Tucker and Fritz John, using new types of the two following most important tools/objects in optimality conditions. The first one is a derivative since it is used to get approximations of the problem data, which is easier than the original ones in dealing with. From the 70s of the last century, nonsmooth problems replace the classical ones with (Fréchet) differentiable data since most practical problems in real life are nonsmooth. However, all nonsmooth results are extensions of the classical ones from the time of Fermat and Lagrange with the central role of derivatives. For generalized derivatives, the reader is referred to comprehensive books [2–6]. The second important tool/object is a type of (relaxed) convexity. The first publication about the role of convexity [7] and the comprehensive books [8,9]. Paper [10], concepts of generalized convexity called quasiconvexity and pseudoconvexity were introduced to prove a new version of the von Neumann minimax theorem in [7]. Naturally, generalized convexity involving types of derivatives has a special role in optimality conditions. Among the types of duality

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research, the Wolfe and Mond-Weir duality schemes connect most closely to optimality conditions.

For the study of optimality conditions together with duality of the Wolfe and Mond-Weir types, as example we can list the publications [11–18].

Motivated by the above observations, to have contributions to the two aforementioned topics with thousands of existing earlier papers in the literature, we focus on developing a little more the tools of generalized derivatives and convexity. Namely, we extend the concept of Gateaux variation and differentiable quasiconvexity proposed [19] in various ways with attention to many other previous contributions. One of the novelties of our generalized quasiconvexity notions is the involved sequences replacing line segments in convexity. We choose the Gateaux variation [19] and relatively far from the known ones. Furthermore, the limit of 8 pages for a paper in the journal does not allow us to try with long developments.

The layout of the paper is simple. After the brief Section ***Preliminaries***, we establish sufficient optimality conditions and necessary ones in Section ***Optimality conditionsin vector optimization*** Sections ***Wolfe duality*** and Section ***Mond-Weir duality*** are devoted to weak and strong duality properties for our Wolfe and Mon-Weir duality schemes. The short Section ***Conclusion*** includes concluding remarks to end the paper.

PRELIMINARIES

In this paper, if not otherwise specified, X,Y,Z are linear spaces (sometimes we assume additionally that is a topological linear space). $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_+$ stand for the set of the natural numbers, the real numbers, and the nonnegative real numbers, respectively (resp). The algebraic interior of A is

$$\begin{aligned} &coreA := \{a \in A | \forall x \in X, \exists \gamma_0 > 0 \\ &such\ as\ a + \gamma x \in A, \forall \gamma \in [0,\gamma_0]\}. \end{aligned}$$

For a linear space, say Y, Y' denotes the algebraic dual of Y and, for $C \subset Y$, C' stands for the positive algebraic dual cone of C, i.e., $C' := \{\lambda \in Y' | \langle \lambda, c\rangle \geq 0,\ \forall c \in C\}$. It is clear that C' is always a convex cone. $\langle \cdot,\cdot\rangle$ signifies the canonical pair between Y and Y'

For an arbitrary set $M \subset X$ and a convex cone $C \subset Y$, $k : M \to Y$ is called -convex-like on M iff $k(M) + C$ is convex. For $f : M \to Y$ and $g : M \to Z$, $(f,g) : M \to Y \times Z$ denotes the map $x \mapsto (f(x), g(x))$. Our problem in this paper is defined now. Let *X, Y, Z* be linear spaces, $S_0 \subset X$ be nonempty, $C_Y \subset Y$ and $C_Z \subset Z$ be nonempty (not necessarily cones), $f : S_0 \to Y$, and $g : S_0 \to Z$. We will consider the (generalized) vector optimization problem

$$(P)\ min_{C_Y} f(x)\ s.t. x \in S_0,\ g(x) \in -C_Z.$$

The feasible set, i.e., the set of points satisfying the constraints of (P), is $S := \{x \in S_0 | g(x) \in -C_Z\}$ and any $\bar{x} \in S$ is called a feasible point for (P).

The concepts of solutions we study in this paper are provided in the following.

Definition 1. (types of minimizers of a map) Let X be topological linear space, Y a linear space, $M \subset X$ nonempty, $C_Y \subset Y$ nonempty, and $k : M \to Y$.

(i) A point $\bar{x} \in M$ is called a *local (Pareto) minimizer* of k on M iff there exists a neighborhood U of $\bar{x}$ such that $k(x) - k(\bar{x}) \notin -C \backslash C$ for all $x \in M \cap U$.

(ii) A point $\bar{x} \in M$ is called a *local strong minimizer* of k on M iff there exists a neighborhood U of $\bar{x}$ such that $k(x) - k(\bar{x}) \notin C_Y$ for all $x \in M \cap U$.

(iii) When $coreC \neq \emptyset$, a point $\bar{x} \in M$ is called a *local weak minimizer* of k on M iff there exists a neighborhood U of $\bar{x}$ such that $k(x) - k(\bar{x}) \notin -coreC$ for all $x \in M \cap U$.

When $U = X$, we delete the word "local" to get the corresponding global concepts. Furthermore, X may be a linear space in this case, not necessarily equipped with a topology.

Of course, a minimizer/solution of (P) is the corresponding minimizer of f on S.

OPTIMALITY CONDITIONS IN VECTOR OPTIMIZATION

In this section, the goal of our study is optimality conditions for problem (P). Both local and global considerations of the aforementioned three types of solutions are concerned. As mentioned in Section 1, in the study of optimality conditions, the most important factors could be the involved generalized derivatives and relaxed convexity. Hence, we specify these two objects we employ in our work now. First, we extend the concept of Gateaux variation [19] as follows for our use.

Definition 2. (Gateaux variation) Let X,Y be linear spaces, $M \subset X$ and $H_1, H_2 \subset Y$ nonempty, $x_0 \in M$ and $k : M \to Y$. A map $k'(x_0) : M - x_0 \to Y$ is called a *Gateaux variation* of k at x_0 with respect to (wrt) H_1, H_2 iff whenever $x \in M \backslash \{x_0\}$ satisfies $k'(x_0)(x - x_0) \in H_2$, then there exist $\bar{t} > 0$ such that $x_t := x_0 + t(x - x_0) \in S_0$ and $\frac{1}{t}(k(x_t) - k(x_0)) \in H_1$ for all $t \in (0;\bar{t}]$.

Definition 3. (generalized quasiconvexity) Let X,Y,Z be linear spaces, $M \subset X$, $x_0 \in M$, $H_1, H_2 \subset Y$ and $K_1, K_2 \subset Y$ nonempty, and $k : M \to Y$.

(i) Assume that k has a Gateaux variation $k'(x_0)$ at wrt H_1, H_2 with $K_1 \subset H_1$ and $K_2 \subset H_2$. k is said to be *on-set differentiably* (K_1, K_2)*-quasiconvex* at x_0 iff whenever $x \in M\setminus\{x_0\}$ satisfies $k(x) - k(x_0) \in K_1$, there exists $\widehat{x} \in M\setminus\{x_0\}$ such that $k'(x_0)(\widehat{x} - x_0) \in K_2$

(ii) Impose the assumption in (i) and additionally that X is a topological linear space. Then, k is called *sequentially differentiably* (K_1, K_2)*-quasiconvex* at x_0 iff from $x_n \in M\setminus\{x_0\} \to x_0$ satisfies $k(x_n) - k(x_0) \in K_1$ for large n, it follows the existence of $\widehat{x} \in M\setminus\{x_0\}$ such that $k'(x_0)(\widehat{x} - x_0) \in K_2$.

In case $K_1 = K_2 =: K$ and/or $H_1 = H_2 =: H$, we write K and/or H in the above definition instead of K_1, K_2 and/or H_1, H_2.

We are now at a position to begin with our first goal (of studying optimality conditions). The first result is the following.

Theorem 1. (sufficient condition for global weak minimizers) Assume that $\bar{x}$ is feasible for (P), core $coreC_Y \neq \varnothing$, $C_Z \subset C_Z + cone(g(\bar{x}))$ and (f,g) has a Gateaux variation $(f'(\bar{x}), g'(\bar{x}))$ at $\bar{x}$ wrt H with $-(coreC_Y \times C_Z) \subset H$. Assume further that

$$\lambda \in (coreC_Y)' \setminus \{0\},\ \mu \in C_Z' \tag{1}$$

such that for all $x \in S_0$,

$$\langle \lambda, f'(\bar{x})(x-\bar{x})\rangle + \langle \mu, g'(\bar{x})(x-\bar{x})\rangle \geq 0 \tag{2}$$

$$\langle \mu, g(\bar{x})\rangle = 0 \tag{3}$$

Then, $\bar{x}$ is a global weak minimizer of problem (P) if the mapping (f,g) is on-set differentiably $-[coreC_Y \times (C_Z + cone(g(\bar{x})))]$-quasiconvex at $\bar{x}$.

Proof Assume that (f,g) is on-set differentiably $-[coreC_Y \times (C_Z + cone(g(\bar{x})))]$-quasiconvex at $\bar{x}$ and relations (1)-(3) are satisfied. We claim that

$$\begin{aligned}&(f'(\bar{x}), g'(\bar{x}))(x-\bar{x}) \notin \\ &-[coreC_Y \times (C_Z + cone(g(\bar{x})))]\end{aligned} \tag{4}$$

for all $x \in S_0\setminus\{\bar{x}\}$. Indeed, reduction at absurdum, suppose that we have a point $\widehat{x} \in S_0\setminus\{\bar{x}\}$ such that the left-hand side of (4) belongs to the right-hand one. Then, there exist $z' \in C_Z$ and $\alpha \geq 0$ such that $f'(\bar{x})(\widehat{x}-\bar{x}) \in -coreC_Y$ and $g'(\bar{x})(\widehat{x}-\bar{x}) = -z' - \alpha g(\bar{x})$. Hence, by (1) and (3),

$$\langle \lambda, f'(\bar{x})(\widehat{x}-\bar{x})\rangle - \langle \mu, z' + \alpha g(\bar{x})\rangle < 0.$$

This contradicts (2). So, the claim (4) is verified. By the assumed on-set differentiable $-[coreC_Y \times (C_Z + cone(g(\bar{x})))]$-quasiconvexity, there does not exist $x \in S_0\setminus\{\bar{x}\}$ such that $f(x) \in f(\bar{x}) - coreC_Y, -g(x) \in C_Z + cone(g(\bar{x}))$. This means that $\bar{x}$ is a global weak minimizer of f on $\widehat{S}$, where

$$\widehat{S} := \{x \in S_0 | g(x) \in -C_Z - cone(g(\bar{x}))\}$$

Observe that $S \in \widehat{S}$ since $C_Z \subset C_Z + con(g(\bar{x}))$. Hence, $\bar{x}$ is a global weak minimizer of f on S, i.e., $\bar{x}$ is a global weak minimizer of problem (P).

Remark 1. (comment on complementarity slackness) The complementarity slackness condition (3), $\langle \mu, \bar{z}\rangle = 0$, is traditionally included in each Karush-Kuhn-Tucker optimality condition. Hence, we keep this equality in the above sufficient condition. In fact, as we see in the proof, this sufficient condition is for a problem with feasible set $\widehat{S}$ larger than S if $C_Z \subset C_Z + cone(g(\bar{x}))$ and so it is also for (P). That is for (P), we can remove the complementarity slackness condition in the above sufficient condition.

Therefore, it is more reasonable not to include slackness conditions in the next statement.

Theorem 2. (the sufficient conditions) Assume that $\bar{x}$ is feasible for (P), core $coreC_Y \neq \varnothing$, $\bar{z} - C_Z \subset -C_Z$ (which is satisfied if C_Z is a convex cone and $\bar{z} \in -C_Z$). Assume further that (1) and (2) hold with a Gateaux variation $(f'(\bar{x}), g'(\bar{x}))$ at $\bar{x}$ wrt H_1, H_2, or H as in Definition 2 but more specified suitably for each assertion below.

(i) $\bar{x}$ is a local weak minimizer of (P) if (f,g) is sequentially differentiably $[-(coreC_Y \times C_Z)]$-quasiconvex at $\bar{x}$ with $(f'(\bar{x}), g'(\bar{x}))$ wrt H satisfying $-(coreC_Y \times C_Z) \subset H$.

(ii) $\bar{x}$ is a local (global, resp) minimizer of (P) if (f,g) is sequentially (on-set, resp) differentiably $[(Y\setminus C_Y) \times (-C_Z), -(coreC_Y \times C_Z)]$-quasiconvex at $\bar{x}$ with $(f'(\bar{x}), g'(\bar{x}))$ wrt H_1, H_2 satisfying $-(coreC_Y \times C_Z) \subset H_2$ and $(-C_Y)\setminus C_Y \times (-C_Z) \subset H_1$.

(iii) $\bar{x}$ is a local (global, resp) strong minimizer of (P) if (f,g) is sequentially (on-set, resp) differentiably $[(Y\setminus C_Y) \times (-C_Z), -(coreC_Y \times C_Z)]$-quasiconvex at $\bar{x}$ with $(f'(\bar{x}), g'(\bar{x}))$ wrt H_1, H_2 satisfying $-(coreC_Y \times C_Z) \subset H_2$ and $(Y\setminus C_Y) \times (-C_Z) \subset H_1$.

Proof. By reasons of similarity, we only work with two types of solutions. First, let us verify the assertion on global minimizes in (ii). Under the imposed assumptions, we check claim (4) for $coreC_Y \times C_Z$ instead of $coreC_Y \times (C_Z + cone(g(\bar{x})))$ in the preceding theorem. Suppose to the contrary that there exists $\widehat{x} \in S_0\setminus\{\bar{x}\}$ such that

$$\left(f'(\bar{x}), g'(\bar{x})\right)(\widehat{x} - \bar{x}) \in -(coreC_Y \times C_Z)$$

Then, (1) implies that

$$\langle \lambda, f(\widehat{x})\rangle + \langle \mu, g(\widehat{x})\rangle < 0$$

contradicting (2). So, the above claim holds. The assumed on-set differentiable quasiconvexity in turn implies that there does not exist $x \in S_0 \backslash \{\bar{x}\}$ such that

$$f(x) \in f(\bar{x}) + (-C_Y)/C_Y$$

and

$$g(x) \in g(\bar{x}) - C_Z \subset -C_Z$$

by assumption, i.e., $\bar{x}$ is a global minimizer of (P).

Next, we sketch the proof for a local strong minimizer in (iii). Like above, the claim (4) also holds in this context. Then, the assumed sequential differentiable quasiconvexity in this case implies that there does not exist $x_n \in S_0 \backslash \{\bar{x}\} \to \bar{x}$ such that $f(x_n) \in f(\bar{x}) + Y \backslash C_Y$ and $g(x_n) \in -C_Z$ for all n, i.e., $\bar{x}$ is a local strong minimizer of (P).

For necessary optimality conditions, we need some new types of generalized quasiconvexity.

Definition 4. (other types of generalized quasiconvexity) Let $M \subset X$,
$H_1,\ H_2,\ K_1,\ K_2 \subset Y$ be nonempty, $x_0 \in M$, and $k : M \to Y$ nonempty-valued.

(i) Assume that k has a Gateaux variation $k'(x_0)$ at x_0 wrt H_1, H_2 with $K_i \subset H_i$, $i = 1, 2.k$ is called *on-set differentiably (K_1, K_2)-quasiconvex of type 2* at x_0 if whenever $\widehat{x} \in M \backslash \{x_0\}$ and $k'(x_0)(\widehat{x} - x_0) \in K_2$ then there exists

$$x \in S_0 \backslash \{x_0\} \textit{ such that } k(x) \in k(x_0) + K_1.$$

(ii) Impose the assumptions in (i) and additionally that X is a topological linear space. k is said to be *sequentially differentiably (K_1, K_2)-quasiconvex of type 2* at x_0 if from the existence of $\widehat{x} \in M \backslash \{x_0\}$ with $k'(x_0)(\widehat{x} - x_0) \in K_2$, it follows that there exists $x_n \in S_0 \backslash \{x_0\} \to x_0$ such that $k(x_n) \in k(x_0) + K_1$ for large n.

In case $K_1 = K_2 =: K$ and/or $H_1 = H_2 =: H$, we apply the convention in Definition 3.

For necessary optimality conditions, we assume additionally that C_Y and C_Z are convex cones with core $C_Z \neq \varnothing$.

Theorem 3. (necessary conditions) Consider problem (P) with the additional assumption given prior to this theorem. Assume that $coreC_Y \neq \varnothing$ and (f, g) has a Gateaux variation $(f'(\bar{x}), g'(\bar{x}))$ at $\bar{x}$ wrt H_1, H_2 as in Definition 2 and more specified for each assertion below.

(i) (for weak minimizers) Assume that $\bar{x}$ is a local (global, resp) weak minimizer of (P). Assume further that (f, g) is sequentially (on-set, resp) differentiably $[-(coreC_Y \times C_Z)]$-quasiconvex of type 2 at $\bar{x}$ with $(f'(\bar{x}), g'(\bar{x}))$ wrt H satisfying $-(coreC_Y \times C_Z) \subset H$. Then, for all $x \in S_0$, there exist $\lambda \in C_Y'$ and $\mu \in C_Z'$ not all zero such that (2) and (3) are satisfied.

If in addition, $(f'(\bar{x}), g'(\bar{x}))$ is $[coreC_Y \times C_Z]$ -convex like on $S_0 - \bar{x}$, then the above (λ, μ) can be chosen common for all $x \in S_0$. Furthermore, if the constraint qualification (CQ): $g'(\bar{x})(S_0 - \bar{x}) = Z$ holds, then $\lambda \neq 0$.

(ii) (for minimizers) The assertions for minimizers are the ones in (i) (for weak minimizers) with sequential (or on-set) differentiable $[(-C_Y) \backslash C_Y \times (-C_Z), -(coreC_Y \times C_Z)]$-quasiconvexity replacing sequential (or on-set, resp) differentiable $[-(coreC_Y \times C_Z)]$-quasiconvexity and with $(f'(\bar{x}), g'(\bar{x}))$ wrt H_1, H_2 satisfying $-(coreC_Y \times C_Z) \subset H_2$ and $(-C_Y) \backslash C_Y \times (-C_Z) \subset H_1$.

(iii) (for strong minimizers) The assertions for strong minimizers are the ones in (i) with sequential (or on-set) differentiable $[Y \backslash C_Y \times (-C_Z), -(coreC_Y \times C_Z)]$-quasiconvexity replacing sequential (or on-set, resp) differentiable $[-(coreC_Y \times C_Z)]$-quasiconvexity and with $(f'(\bar{x}), g'(\bar{x}))$ wrt H_1, H_2 satisfying $-(coreC_Y \times C_Z) \subset H_2$ and $(Y \backslash C_Y) \times (-C_Z) \subset H_1$.

Proof. By reasons of similarity, we only prove the case of a local weak minimizer in (i).

Consider first the claim (4) (with $coreC_Y \times C_Z$ instead of $coreC_Y \times (C_Z - cone(g(\bar{x})))$ by contradiction. Suppose the existence of $\widehat{x} \in S_0$ such that $f(\widehat{x}) \in -coreC_Y$ and $g(\widehat{x}) \in -C_Z$. Then, $\langle \lambda, f(\widehat{x}) \rangle + \langle \mu, g(\widehat{x}) \rangle < 0$.

By the sequential differentiable $[-(coreC_Y \times C_Z)]$-quasiconvexity of type 2, there exists $x_n \in S_0 \backslash \{\bar{x}\} \to \bar{x}$ such that $f(x_n) \in f(\bar{x}) - coreC_Y$ core and $g(x_n) \in -C_Z$ for large n, contradicting the local weak minimality of $\bar{x}$. So, the claim holds.

Since the convex set $-(coreC_Y \times C_Z)$ has nonempty algebraic interior, applying the separation theorem paper[20], we can separate this set and the point $(f'(\bar{x}), g'(\bar{x}))(x - \bar{x})$ to obtain multipliers $\lambda \in Y'$ and $\mu \in Z'$ not all zero satisfying (2). As C_Y and C_Z are convex cones, we see that $\lambda \in C_Y'$ and $\mu \in C_Z'$ with $\langle \mu, g(\widehat{x}) \rangle = 0$ as (3) required.

Under the additional convex-likeness, we apply the aforementioned separation theorem to separate two convex sets $(f'(\bar{x}), g'(\bar{x}))(S_0 - \bar{x}) + coreC_Y \times C_Z$ and $-(coreC_Y \times C_Z)$ to get the common multipliers (λ, μ) as required. Suppose that $\lambda = 0$. Then, for all $x \in S_0$, $\langle \mu, g(\bar{x})(x - \bar{x}) \rangle \geq 0$. Take arbitrary $\widehat{z} \in Z$. In view of (CQ), there exists $\widehat{x} \in S_0$ such that $\widehat{z} = g'(\bar{x})(\widehat{x} - \bar{x})$ and $\langle \mu, \widehat{z} \rangle \geq 0$. By the arbitrariness of $\widehat{z} \in Z$, μ must be zero, which is impossible as λ is supposed to be zero.

WOLFE DUALITY

A duality scheme [11], called later the Wolfe duality, quite different from the earlier well-known important Lagrange duality model, was introduced. This concept has been developed also until now alongside with the Lagrange and the Mond-Weir [13] schemes and also others such as the one induced from the dual pair of Stampacchia and Minty variational inequalities, etc. Here, together with (P), we investigate the following Wolfe-type dual problem, for a fixed $e \in coreC_Y$,

$$(D_W) \quad \max_{C_Y}(f(u) + \langle \mu, g(u)\rangle e) u \in S_0, \\ \lambda \in C_Y' \backslash \{0\}, \mu \in C_Z', e \in \text{core}\, C_Y \langle \lambda, e\rangle = 1 \tag{5}$$

$$\left\langle \lambda, f'(u)\left(u' - u\right)\right\rangle + \\ \left\langle \mu, g'(u)\left(u' - u\right)\right\rangle \geq 0, \ for\ all\ u' \in S_0 \tag{6}$$

Here $(f'(u), g'(u)) : S_0 - u \to Y \times Z$ is a Gateaux variation of (f, g) (following Definition 2) with details specified in each consideration context below. Maximizing over C_Y can mean finding maximal solutions in both local and global considerations. A point (u, λ, μ) satisfying relations (5) and (6) is called a feasible point for problem (D_W).

For our duality study, we need the following generalized quasiconvexity.

Definition 5. (types of scalar generalized convexity) Let $S_0 \subset X$, $u \in S_0$, $k : S_0 \to Y \times Z$, $k'(u) : S_0 - u \to Y \times Z$ a Gateaux variation of k at u wrt any H_1, H_2 (following Definition 2, but in this case $H_1, H_2 \subset Y \times Z$).

(i) (scalar generalized quasiconvexity) k is called *scalar generalized quasiconvex* at u if from $x \in S_0$, $(y,z) = k(x)$, $(v,w) = k(u)$, $\lambda \in C_Y' \backslash \{0\}$, and $\mu \in C_Z'$ satisfying (5), it follows that there exist $\lambda' \in C_Y' \backslash \{0\}$ and $u' \in S_0$ such that, for $(v', w') = k'(u)(u' - u)$,

$$\langle \lambda', y - v\rangle + \langle \mu, z - w\rangle \geq \langle \lambda, v'\rangle + \langle \mu, w'\rangle.$$

(ii) (scalar generalized strict convexity) k is called *scalar generalized strictly convex* at u if for all $x \in S_0$ with $(y,z) = k(x)$, $(v,w) = k(u)$, $\lambda \in C_Y' \backslash \{0\}$, and $\mu \in C_Z'$, there exist $\lambda' \in C_Y' \backslash \{0\}$ and $\mu' \in S_0$ such that, for $(v', w') = k(u)(u' - u)$,

$$\langle \lambda', y - v\rangle + \langle \mu, z - w\rangle > \langle \lambda, v'\rangle + \langle \mu, w'\rangle.$$

Theorem 4. (Wolfe weak duality) Assume that core$C_Y \neq \varnothing$, x and (u, λ, μ) are feasible for (P) and (D_W), resp, and (f, g) has a Gateaux variation $(f'(u), g'(u))$ at u specified suitably in each assertion below.

(i) If (f, g) is scalar generalized quasiconvex at u with $(f'(u), g'(u))$ wrt H_1, H_2 satisfying $-(coreC_Y \times C_Z) \subset H_2$ and $-(coreC_Y \times C_Z) \subset H_1$, then

$$f(x) \notin f(u) + \langle \mu, g(u)\rangle e - coreC_Y.$$

(ii) If (f, g) is scalar generalized strictly convex at u with $(f'(u), g'(u))$ wrt H_1, H_2 satisfying $-(coreC_Y \times C_Z) \subset H_2$ and $(-C_Y)/C_Y \subset H_1$, then

$$f(x) \notin f(u) + \langle \mu, g(u) w\rangle e + (-C_Y) \backslash C_Y.$$

Proof. (i) Suppose to the contrary that

$$f(x) - f(u) - \langle \mu, g(u)\rangle e \in -coreC_Y.$$

In virtue of the assumed scalar generalized quasiconvexity of (f, g), there exist $\lambda' \in C_Y' \backslash \{0\}$ and $\widehat{u} \in S_0$ with $(\widehat{v}, \widehat{w}) = (f'(u), g'(u))(\widehat{u} - u)$ such that

$$\langle \lambda', f(x) - f(u)\rangle + \langle \mu, g(z) - g(u)\rangle \\ \geq \langle \lambda, \widehat{v}\rangle + \langle \mu, \widehat{w}\rangle.$$

As $\langle \mu, g(x)\rangle \leq 0$ by (5), from the above contradiction assumption, it follows that

$$0 > \langle \lambda', f(x)\rangle - \langle \lambda', f(u)\rangle - \langle \mu, g(u)\rangle \\ \geq \langle \lambda', f(x)\rangle + \langle \mu, g(x)\rangle - \langle \lambda, f(u)\rangle - \langle \mu, g(u)\rangle \\ = \langle \lambda', f(x) - f(u)\rangle + \langle \mu, g(x) - g(u)\rangle.$$

This is a contradiction to (6).

(ii) Suppose that

$$f(x) \in f(u) + \langle \mu, g(u)\rangle e + (-C_Y) \backslash C_Y.$$

Then, in view of the assumed scalar generalized strict convexity of (f, g), as $\langle \mu, g(x)\rangle \leq 0$ there exist $\lambda' \in C_Y' \backslash \{0\}$ and $\widehat{u} \in S_0$ such that, for $(\widehat{v}, \widehat{w}) = (f'(u), g'(u))(\widehat{u} - u)$,

$$0 \geq \langle \lambda', f(x) - f(u)\rangle + \langle \mu, g(x) - g(u)\rangle \\ \geq \langle \lambda, \widehat{v}\rangle + \langle \mu, \widehat{w}\rangle,$$

which also contradicts (6).

Theorem 5. (Wolfe strong duality) Impose the assumptions of Theorem 3, including the ones in assertions (i)-(iii).

(i) Assume that $\bar{x}$ is a local or global weak minimizer of (P) and (f, g) is scalar generalized quasiconvex at $\bar{x}$ with $(f'(\bar{x}), g'(\bar{x}))$ as in Theorem 4(i) (with $\bar{x}$ replacing u). Then, there exists $\left(\bar{\lambda}, \bar{\mu}\right) \in C_Y' C_Z'$ such that $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ is a global weak maximizer of (D_W).

(ii) Assume that $\bar{x}$ is a local or global minimizer of (P) and (f, g) is scalar generalized strictly convex at $\bar{x}$ with $(f'(\bar{x}), g'(\bar{x}))$ as in Theorem 4(ii) (with $\bar{x}$ replacing u). Then, there exists $\left(\bar{\lambda}, \bar{\mu}\right) \in C_Y' C_Z'$ such that $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ is a global maximizer of (D_W).

Proof. (i) Assume that $\bar{x}$ is a local or global weak minimizer of (P). By Theorem 3(i), there exist $\bar{\lambda} \in C_Y'$ and

$\bar{\mu} \in C'_Z$ not all zero such that (2) and (3) hold for all $x \in S_0$. Hence, (5) and (6) are satisfied with $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ in the place of (u, λ, μ), i.e., $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ is feasible for (D_W).

By the assumed scalar generalized quasiconvexity, one has the weak duality given in Theorem 4(i). Suppose that $\left(\bar{x}, \bar{\lambda}, \bar{\mu}\right)$ is not a global weak maximizer of (D_W), i.e., there exists a feasible point (u, λ, μ) of (D_W) such that

$f(u) + \langle \mu, g(u) \rangle e \in f(\bar{x}) + \langle \bar{\mu}, g(\bar{x}) \rangle e + coreC_Y$

Since $\langle \bar{\mu}, g(\bar{x}) \rangle = 0$ by (3), this means that $f(\bar{x}) \in f(u) + \langle \mu, g(u) \rangle e - coreC_Y$ and contradicts the weak duality in Theorem 4(i). The proof of (ii) is similar, using assertion (ii) in Theorem 4.

MOND-WEIR DUALITY

The Mond-Weir duality scheme [13] was proposed and also intensively developed like the Wolfe one. Let X, Y, Z,S_0, C_Y, C_Z, f, g, and f', g' be as in problem (D_W). Assume additionally that $0 \in -C_Z$. Here, we define the Mond-Weir dual problem of (P) as follows.

$$\max\nolimits_{C_Y} f(u), \quad u \in S_0, \lambda \in C'_Y \backslash \{0\}, \mu \in C'_Z \tag{7}$$

$$\begin{aligned}&\left\langle \lambda, f\left(u'\right)\left(u' - u\right)\right\rangle + \\ &\quad \left\langle \mu, g\left(u'\right)\left(u' - u\right)\right\rangle \geq 0 \; for\; all\; u' \in S_0\end{aligned} \tag{8}$$

$$\langle \mu, g(u) \rangle \geq 0 \tag{9}$$

Here $(f'(u), g'(u)) : S_0 - u \to Y \times Z$ is a mapping associated with (f, g) and specified in the study situations for (D_{MW}). Maximizing wrt C_Y may signify looking for maximal points on the feasible set defined by (7)-(9) in both local and global considerations, depending on the consideration context. A point (u, λ, μ) satisfying relations (7)-(9) is said to be a feasible point for (D_{MW}).

Theorem 6. (Mond-Weir weak duality) Assume that core $C_Y \neq \emptyset$, x and (u, λ, μ) are feasible for (P) and (D_{MW}), resp, and (f, g) has a Gateaux variation $(f'(u), g'(u))$ specified suitably in each assertion below.

(i) Under the assumption of (i) in Theorem 4, one has

$$f(x) \notin f(u) - coreC_Y$$

(ii) The same assumption in Theorem 4 (ii) implies that

$$f(x) \notin f(u) - (-C_Y) \backslash C_Y.$$

Proof. (i) Suppose that $f(x) \in f(u) - coreC_Y$. By the assumed scalar generalized quasiconvexity, there exist $\lambda' \in C'_Y \backslash \{0\}$ and $u' \in S_0$ such that, for $(v', w') = (f'(u), g'(u))(u' - u)$,,

$$\begin{aligned}&\langle \lambda', f(x) - f(u) \rangle + \langle \mu, 0 - g(u) \rangle \\ &\geq \langle \lambda, v' \rangle + \langle \mu, w' \rangle.\end{aligned}$$

But $0 > \langle \lambda', f(x) - f(u) \rangle + \langle \mu, 0 - g(u) \rangle$. Hence, $0 > \langle \lambda, v' \rangle + \langle \mu, w' \rangle$, contradicting (8).

(ii) Suppose to the contrary that $f(x) \in f(u) + (-C_Y) \backslash C_Y$. Then, by the assumed scalar generalized strict convexity of (f, g), as $\langle \mu, g(x) \rangle \leq 0$ and $\langle \mu, g(u) \rangle \geq 0$, there exist $\lambda' \in C'_Y \backslash \{0\}$ and $\widehat{u} \in S_0$ such that, for $(\widehat{v}, \widehat{w}) = (f'(u), g'(u))(\widehat{u} - u)$,

$$\begin{aligned}&0 \geq \langle \lambda', f(x) - f(u) \rangle + \langle \mu, g(x) - g(u) \rangle \\ &> \langle \lambda, \widehat{v} \rangle + \langle \mu, \widehat{w} \rangle,\end{aligned}$$

which also contradicts (8).

By arguments similar to the proof of Theorem 5, we obtain the following corresponding strong duality statement.

Theorem 7. (Mond-Weir strong duality) The strong duality relations in Theorem 5 for (D_W) are valid also for (D_{MW}).

CONCLUSIONS

In this paper, optimality conditions together with Wolfe and Mond-Weir duality properties are studied for a general vector optimization problem. The main characteristic features here are the following. The problem setting is general with linear underlying spaces for most cases. The "orderings" in the objective and constraint spaces in a part of the results are defined by arbitrary nonempty sets, not necessarily convex cones. Generalized derivatives and convexity, the two main factors in any optimality conditions and duality results, are proposed for the use in the paper and remarkably different from almost all the corresponding concepts employed in the earlier contributions we know. The novelties and advantages of our results are clear enough.

For possible perspectives, we think that the results here can be developed for some recent models attracting attention from many researchers such as set-valued problems, vector problems with variable preferences, and set optimization, etc.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interest in publishing this paper.

AUTHORS' CONTRIBUTIONS

This study is finished by one author.

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Bài nghiên cứu

Điều kiện tối ưu và đối ngẫu với biến phân mới của đạo hàm suy rộng và tính lồi

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TÓM TẮT

Bài toán tối ưu vector tổng quát với các ràng buộc bất đẳng thức được xem xét, chủ đề này là mô hình rất phổ biến và quan trọng trong lĩnh vực tối ưu hóa. Tính tổng quát của của việc thiết lập chủ yếu được thể hiện ở ba yếu tố sau. Các không gian được xét là không gian tuyến tính không cần topo (ngoại trừ không gian quyết định được trang bị thêm với cấu trúc này cho một số kết quả). "Thứ tự" trong cả không gian mục tiêu và không gian không gian của các ràng buộc được xác định bởi các tập tùy ý khác rỗng (không nhất thiết là nón lồi). Dữ liệu bài toán là ánh xạ không trơn, tức là chúng không khả vi Fréchet. Trong bài này điều kiện tối ưu và đối ngẫu của Wolfe và Mond-Weird được xem xét đây là vấn đề trọng tâm của lý thuyết tối ưu. Các kết quả này được thiết lập cho ba loại nghiệm: nghiệm Pareto, nghiệm yếu và nghiệm mạnh với nghiệm địa phương và toàn cục. Trong nội dung về điều kiện tối ưu, giới thiệu về tựa lồi khả vi trên tập cho nghiệm toàn cục và tựa lồi khả vi theo dãy cho trường hợp địa phương. Hơn thế nữa, mỗi khái niệm gồm loại 1 và 2, điều xét điều kiện cần và đủ được xét cho loại 1 và loại 2 nêu trên. Sau khi đạt được điều kiện tối ưu, áp dụng điều kiện tối ưu để được đối ngẫu mạnh và đối ngẫu yếu cho các loại nghiệm vào mô hình của hai loại đối ngẫu Wolfe và Mon-Weir. Do tính phức tạp của chủ đề: xét đối ngẫu là khác nhau từ những điều kiện tối ưu, chúng tôi đã xét nhiều hơn 2 loại tựa lồi suy rộng: tựa lồi vô hướng cho nghiệm yếu và lồi vô hướng chặt cho nghiệm Pareto. Vì vậy tất cả các kết quả được đề cập về biến phân Gateaux và những dạng tựa lồi suy rộng khác. Các kết quả là nổi bật với những kết quả đã biết với những phát triển trong những trường hợp ứng dụng đặc biệt.

Từ khoá: Bài toán tối ưu vector tổng quát, Biến phân Gateaux, Tựa lồi khả vi theo dãy, Tựa lồi khả vi trên tập, Điều kiện cần, Điều kiên đủ, Tựa lồi suy rộng vô hướng, Lồi chặt suy rộng vô hướng, Đối ngẫu Wolfe, Đối ngẫu Mond-Weir

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