Characterizations of variational convergence of bifunctions defined on products of two sets

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ABSTRACT
We present definitions of types of variational convergence of finite-valued bifunctions defined on rectangular domains and establish characterizations of these convergences. In the introduction, we present the origins of the research on variational convergence and then we lead to the specific problem of this paper. The content of the paper consists of 3 parts: variational convergence of function; variational convergence of bifunction; and characterizations of variational convergence of bifunction, this part is the main results of this paper. In section 2, we presented the definition of epi-convergence and presented a basic property problem that will be used to extend and develop the next two sections. In section 3, we start to present a new definition, the definition of convergence epi / hypo, minsup and maxinf. To clearly understand of these new definitions we have provided comments (remarks) and some examples which reader can check these definitions. The above contents serve the main result of this paper will apply in part 4. Now, we will explain more detail for this part as follows. Firstly, variational convergence of bifunctions is characterized by the epi- and hypo-convergence of related unifunctions, which are slices sup- and inf-projections. The second characterization expresses the equivalence of variational convergence of bifunctions and the same convergence of the so-called proper bifunctions defined on the whole product spaces. In the third one, the geometric reformulation, we establish explicitly the interval of all the limits by computing formulae of the left- and right-end limit bifunctions, and this is necessary and sufficient conditions of the sequence bifunctions to attain epi / hypo, minsup and maxinf convergence.

Key words: Variation convergence, Epi-convergence, Hypo-convergence, Epi/hypo-convergence, Lopsided convergence, Equivalence class

INTRODUCTION
Epi-convergence, epi/hypo-convergence, and lopsided convergence are the main types of variational convergence of unifunctions (briefly, functions) and bifunctions, respectively. Epi-convergence was introduced independently 1–3. For details of this convergence and its applications, the reader is referred to the books 4–6.

Epi/hypo-convergence of extended-real-valued bifunctions was proposed and developed 7,8. Typical references for this convergence and its applications, were first studied 9. After the appearance 7, lopsided convergence, a modified version stronger than epi/hypo-convergence was further studied 10. All the aforementioned contributions investigate the class of extended-real-valued functions defined on the whole space X, denoted by fcn (X), or of bifunctions defined on the whole product space X Y denoted by biv (X Y). Lopsided convergence was discussed for the first time for a class of finite-valued bifunctions defined on A × B, where A ⊂ X and B ⊂ Y with X, Y being finite-dimensional spaces 7.

This class is also called finite-valued bifunctions on rectangles and designated by fv-biv (X Y). It is important because typical bifunctions in many mathematical models of practical problems are Lagrange functions 14–16 in constrained optimization, Hamilton functions in variational calculus and optimal control, and payoff functions in zero-sum games, all belong to this class. Epi/hypo-convergence of finite-valued bifunctions on rectangles was studied 16–18. Lopsided convergence for special subclasses of fv-biv (X Y) was investigated 18–20. Lopsided convergence is stronger than epi/hypo-convergence. Hence, it has more beautiful properties. But, it is not symmetric with respect to the two components x ∈ X and y ∈ Y. It has “one-sided” properties and is appropriated for either minsup- or maxinf-properties. However, the first variational properties of bifunctions attracting attention from researchers must be saddle-point properties 21–22, which combine both the minsup- and maxinf-properties. Especially, such properties are crucial for studies of dualities in general and dual problems 23 in particular.

further that recently lopsided convergence of finite-valued bifunctions and its applications were intensively studied for special types of bifunctions with applications mainly in special important classes of problems. However, for epi/hypo-convergence of finite-valued bifunctions, we see only papers for the finite-dimensional case.

Motivated by the above brief about results on variational convergence obtained up to this time, in this paper we study variational convergence of finite-valued bifunctions on rectangles in the case of general metric spaces. Our aim is characterizations of such a convergence. The obtained results are new in various aspects or improve/extend the corresponding ones for the finite-dimensional case.

Namely, we develop three characterizations of the epi/hypo- and lop-convergence of bifunctions in f-vbiv in the general metric-space case. It should be stressed that characterizations of variational convergence are important not only for theoretical developments but in fact even more crucial for applications.

In this paper, all the spaces are metric spaces, if not otherwise specified. Our notation is standard. For instance, i, j, and k stand for the real line, the extended real line \( \mathbb{R} \), the set of the natural numbers, respectively. For \( A \subseteq X \), int \( A \) and bd \( A \) denote the interior and the boundary of \( A \), resp. Function \( \varphi : X \rightarrow \mathbb{R} \) is called lower semicontinuous (lsc, upper semicontinuous (usc)) at \( x \) if \( \liminf_{y \to x} \varphi(y) \geq \varphi(x) \) (\( \limsup_{y \to x} \varphi(y) \leq \varphi(x) \), resp.). For \( A \subseteq X \), the lower/inferior limit and the upper/outer limit are defined by

\[
\liminf_{x \to a^{-}} \varphi(x) = \{ x \in X : \liminf_{y \to x} \varphi(y) = x \},
\]

\[
\limsup_{x \to a^{+}} \varphi(x) = \{ x \in X : \limsup_{y \to x} \varphi(y) = x \}.
\]

If \( \liminf_{x \to a^{-}} A = \limsup_{x \to a^{+}} A \), we say that \( A \) tend to \( a \) in the Painlevé–Kuratowski sense, denoted by \( A \rightarrow_{p-K} a \) if for any \( x \rightarrow a \) in \( A \), then \( x \) is a minimizer of \( \varphi \).

**Definition 1** (epi-convergence) \( \{ \varphi^k \}_k \) in f-vfcn \((X)\) is said to epi-converge to \( \varphi \) if \( \varphi = e - \lim_k \varphi^k \), if

\( (a) \) for all \( x^k \in A^k \rightarrow x \), \( x^k \), \( \varphi^k(x^k) \geq \varphi(x) \) when \( x \in A \) and \( \varphi^k(x^k) \rightarrow +\infty \) when \( x \notin A \); \n
\( (b) \) for all \( x \in A \), there exist \( x^k \in A^k \rightarrow x \) such that \( \lim_k \varphi^k(x^k) \leq \varphi(x) \).

**Definition 2** (lower and upper epi-limits) Let \( \varphi^k \in \text{fv-fcn} (X) \) and \( x \in X \). The lower epi-limit of \( \{ \varphi^k \}_k \) at \( x \) is \( \text{eli} \varphi^k(x) := \inf \{ \varphi^k(x^k) : x^k \in A^k \} \), where \( \{ x^k \in A^k \rightarrow x \} \) means that we consider all the possible sequences \( x^k \in A^k \) tending to \( x \).

The upper epi-limit of \( \{ \varphi^k \}_k \) at \( x \) is \( \text{els} \varphi^k(x) := \inf \{ \varphi^k(x^k) : x^k \in A^k \} \).

It can be proved that \( \varphi^k \rightharpoonup \varphi \) if and only if \( \text{eli} \varphi^k(x) = \text{els} \varphi^k(x) = \varphi(x) \) when \( x \in A \) and \( \text{eli} \varphi^k(x) = + \) when \( x \notin A \).

The notion symmetric to epi-convergence is hypo-convergence which is defined as follows. \( \varphi^k \) are called hypo-convergent to \( \varphi \), designated by \( \varphi = h - \lim_k \varphi^k \) or \( \varphi^k \rightharpoonup h \varphi \), if \( \varphi^k \rightharpoonup - \varphi \) epi-converge to \(-\varphi \). Hence, the lower and upper hypo-limits of \( \{ \varphi^k \}_k \) at \( x \in A \), resp.,

\[
\text{hli} \varphi^k(x) := \sup \{ \varphi^k(x^k) : x^k \in A^k \},
\]

\[
\text{hls} \varphi^k(x) := \sup \{ \varphi^k(x^k) : x^k \in A^k \}.
\]

Similar to epi-convergence, \( \varphi^k \rightharpoonup h \varphi \) if and only if \( \text{hli} \varphi^k(x) = \text{hls} \varphi^k(x) = \varphi(x) \) when \( x \in A \) and \( \text{hli} \varphi^k(x) = - \) when \( x \notin A \).

Now we extend to the metric-space case the basic variational property of epi-convergence, proved for the finite-dimensional case.

**Theorem 1** (basic variational property of epi-convergence) \( \varphi^k \), \( \varphi \in \text{fv-fcn} (X) \) with \( \varphi = e - \lim_k \varphi^k \), one has \( \text{ls}_k \{ \text{inf}_{x \in A^k} \varphi^k(x) \} \). Moreover, if \( x^k \) is a minimizer of \( \varphi^j \) for all \( j \in \mathbb{N} \) and \( x^k \rightarrow x \in A \), then \( x \) is a minimizer of \( \varphi \) and the infimal value of \( \varphi^k \) also converge to the infimal value of \( \varphi \).

The proof is similar to the known proof for the finite-dimensional case and so omitted. The second part can be reformulated equivalently as follows: if \( e - \lim_k \varphi^k = \varphi \), then

\[
A \subseteq \text{ls}_k (\text{argmin}_{x \in A} \varphi^k) \parallel \text{argmin}_{x \in \mathbb{R}} \varphi.
\]

one can prove easily an extension of this inclusion: if \( e^k \), then

\[
A \subseteq \text{ls}_k (e^k - \text{argmin}_{x \in \mathbb{R}} \varphi^k) \parallel \text{argmin}_{x \in \mathbb{R}} \varphi.
\]

To have the reverse inclusion with the full Lim replacing Limsup together with the convergence of the corresponding infimal values, one needs the following tightness condition.

**VARIATIONAL CONVERGENCE OF FUNCTIONS**

In this section, we present epi-convergence of functions and its basic variational properties. Let \( X \) be a metric space, \( A^k, A \subseteq X \) be nonempty sets, and \( \{ \varphi^k : A^k \rightarrow [0, \infty) \} \) be elements of f-vfcn \((X)\).

The following definition extends the corresponding notion from the case of finite-dimensional spaces to that of general metric spaces.

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Definition 3 (tight epi-convergence) one says that \( \{ \phi^k \}_{k} \) epi-converge tightly to \( \phi \) in \( \text{fv-fcn} (X) \) if they epi-converge and for each \( \varepsilon > 0 \), there exists a compact set \( B \) in \( X \) and \( k_\varepsilon \in \mathbb{N} \) such that, for all \( k \geq k_\varepsilon \), \( \inf_{\text{int}_B \text{A}} \phi^k + \varepsilon \)

Theorem 2 (basic property of tight epi-convergence) \( \phi^k \) tightly epi-converges to \( \phi \) and \( \text{int}_A \phi \) is finite, then \( \text{int}_A \phi^k \supseteq \text{int}_A \phi \). The converse holds if \( X \) is finite-dimensional.

(ii) Assume that \( \phi^k \) epi-convergent to \( \phi \) and \( \text{int}_A \phi \) is finite. If there exist \( x^k \xrightarrow{\varepsilon} 0 \) such that \( \lim_{k} \phi^k - \arg\min_{A} \phi^k = \arg\min_{A} \phi \), then \( \phi^k \) epi-tightly converges to \( \phi \). The converse is true if \( X \) is separable.

To prove this statement, modify suitably the arguments for the finite-dimensional case and apply the theorem on level sets associated with epi-convergence. Due to the requirement of a limited length of the paper, we skip the details.

**VARIATIONAL CONVERGENCE OF BIFUNCTIONS**

For \( A, A^k \in X, B, B^k \subseteq Y, \phi^k : A^k \times B^k \rightarrow \mathbb{R} \), and \( \phi : A \times B \rightarrow \mathbb{R} \), we propose the following definition.

Definition 4 (epi/hypo-convergence) Bifunctions \( \phi \) in \( \text{fv-biv} (X \times Y) \) are called epi/hypo-convergent (e/h-convergent) \( \phi \) to \( \phi \) in \( \text{fv-biv} (X \times Y) \) if

(a) for all \( \phi^k \in B^k \) such that \( \inf_{\text{int}_B \phi} \phi^k = \inf_{\text{int}_B \phi} \phi \)
(b) for all \( \phi^k \in B^k \) such that \( \inf_{\text{int}_B \phi} \phi^k = \inf_{\text{int}_B \phi} \phi \)

This convergence is denoted by \( \phi \xrightarrow{e/h} \phi \).

Note that if \( \phi^k \) do not depend on \( x \), epi/hypo-convergence becomes epi-convergence, and if they do not depend on \( x \), it collapses to hypo-convergence.

However, the epi/hypo-convergence of \( \phi (x, \cdot) \) is not the epi-convergence of \( \phi (\cdot, y) \) for all \( y \)

This is a sufficient condition for e/h-convergence, but not a necessary condition. We see that e/h-convergence and h/e-convergence are symmetric. Moreover, h/e-convergence coincides with e/h-convergence, if one keeps taking minimum on \( x \) and maximum on \( y \), only changes the order of the two operations. The lopsided convergence defined below is our extension to the metric-space case of the corresponding concept for the finite-dimensional case.

Definition 5 (minsup-lop convergence) \( \phi^k \in \text{fv-lsup} (X \times Y) \) are called minsups-lopsided (minsup-lop) convergent to \( \phi \) in \( \text{fv-lsup} (X \times Y) \) if

(a) for each \( y \in B \) and \( x \in A^k \rightarrow x \), there exist \( x^k \in B^k \rightarrow y \) such that \( \inf_{y} \phi^k (x^k, y^k) \leq \phi (x, y) \) when \( x \in A \) and \( \phi^k (x^k, y^k) \rightarrow +\infty \) when \( x \notin A \);

(b) for all \( x \in A \), there exist \( x^k \in A \rightarrow x \) such that, for all \( x^k \in B^k \rightarrow y \) such that \( \inf_{y} \phi^k (x^k, y^k) \leq \phi (x, y) \) when \( y \in B \) and \( \phi^k (x^k, y^k) \rightarrow +\infty \) when \( y \notin B \).

This convergence is denoted by \( \phi \xrightarrow{\text{minsup-lop}} \phi \).

Clearly the roles of \( x \) and \( y \) are not symmetric in this definition. Changing the order of \( x \) and \( y \) leads to the following definition of maxinf-lop convergence:

(a) for all \( x \in A \) and \( y \in B^k \rightarrow y \), there exist \( x^k \in A^k \rightarrow x \) such that \( \inf_{y} \phi^k (x^k, y^k) \leq \phi (x, y) \) when \( x \in B \) and \( \phi^k (x^k, y^k) \rightarrow +\infty \) when \( y \notin B \);

(b) for all \( x \in A \) and \( x^k \in A^k \rightarrow x \), there exist \( x^k \in B^k \rightarrow y \) such that, for all \( x^k \in A^k \rightarrow x \), \( \inf_{y} \phi^k (x^k, y^k) \leq \phi (x, y) \) when \( x \in A \) and \( \phi^k (x^k, y^k) \rightarrow +\infty \) if \( x \notin A \).

We denote this convergence by \( \phi \xrightarrow{\text{maxinf-lop}} \phi \).

It is obvious that each minsup-lorp or maxinf-lop convergence implies e/h-convergence. Condition (a) of e/h-convergence and minsup-lop convergence are the same. While condition (b) of lorp-convergence is stronger (b) of e/h-convergence. Indeed, for \( x \notin A \) lorp-convergence requires the existence of a common sequence \( x^k \rightarrow x \) such that for all \( x^k \in B^k \rightarrow y \), \( \inf_{y} \phi^k (x^k, y^k) \leq \phi (x, y) \) when \( y \in B \) and \( \phi^k (x^k, y^k) \rightarrow +\infty \) when \( y \notin B \).

Clearly, this sequence does not satisfy (b) of e/h-convergence. However, conversely, if one has \( \phi^k \xrightarrow{e/h} \phi \), one still cannot derive that \( \phi^k \) minsups-lorp converges to \( \phi \) as shown by the following example.

**Example 1** Let \( A^k = [1/k, 1] \), \( A = [0, 1] \), and

\[
\phi^k (x, y) = \begin{cases} 
1 & \text{if } (x, y) \in A^k \times B^k, x \neq x^k, y \neq y^k, \\
0 & \text{if } (x, y) \in A \times B^k, x = x^k, y \neq y^k, \\
1 & \text{if } (x, y) \in A^k \times B^k, x \neq x^k, y \neq y^k. 
\end{cases}
\]

Then,

\[
\phi (x, y) = \begin{cases} 
1 & \text{if } (x, y) \in [0, 1]^2, x \neq x^k, y \neq y^k, \\
0 & \text{if } (x, y) \in [0, 1]^2, x = x^k, y \neq y^k, \\
1 & \text{if } (x, y) \in [0, 1]^2, x \neq x^k, y \neq y^k. 
\end{cases}
\]

Clearly \( \phi^k \xrightarrow{e/h} \phi \). We show that condition (b) of minsups-lorp convergence is violated. For \( x = 0 \) and any \( x^k \rightarrow x \), we take \( y = 0 \) and \( y^k \rightarrow 0 \) such that \( x^k = x \) for all \( k \). Then, \( \inf_{y} \phi^k (x^k, y^k) = 1 \geq \phi (x, y) \).
max-inf-lip convergences. (Furthermore, the continuous convergence of $\phi$ implies also both the epiconvergence and hypocovergence of $\phi^k$.) Hence, continuous convergence is a type of variational convergence. But, it is difficult to be satisfied. (ii) Similar to the paper\textsuperscript{16} for the finite-dimensional case, we show in Theorem 5 that in our general case $e/h$-limits are not unique, but form (epi/hypo) equivalence classes. Fortunately, all the bifunctions in such a class have the same variational properties.

Note that the above definitions of variational convergence of bifunctions do not require that $A^k \overset{\text{f}}{\to} B^k \to A \times B$. That is why not only points $(x, y) \in A \times B$ are under consideration, but also all the points $(x, y)$, which are limits of $(x^k, y^k) \in A^k \times B^k$, are taken into account.

In order to see the relation between $fv$-biv $(X^0, Y^0)$ and biv $(X, Y)$, given $\phi$ in $fv$-biv $(X^0, Y^0)$, we define the corresponding two so-called proper bifunctions in biv $(X, Y)$ as follows.

$$
(\eta_{e,h} \phi)(x, y) := \begin{cases} 
\phi(x, y) & \text{if } (x, y) \in A^e \times B^h \\
+ & \text{if } y \notin B, x \notin A, \\
- & \text{if } x \notin B, y \notin A.
\end{cases}
$$

$$
(\eta_{e,h} \phi)(x, y) := \begin{cases} 
\phi(x, y) & \text{if } (x, y) \in A^e \times B^h \\
+ & \text{if } y \notin B, x \notin A, \\
- & \text{if } x \notin B, y \notin A.
\end{cases}
$$

Then, we have in fact two “projections” $\eta_{e,h}$ and $\eta_{h/e}$ from $fv$-biv $(X^0, Y^0)$ to biv $(X, Y)$ transforming $\phi$ into the two corresponding proper bifunctions belonging to biv $(X, Y)$. Looking at the definition of $e/h$-convergence, we see that beside $(x, y) \in A \times B$, we consider also points $(x, y)$ with either $x \in A$ and $y \notin B$ or $x \notin A$ and $y \in B$. We call the points $(x, y) \in A \times B$ and these points the related points (through $e/h$-convergence). For all the related points, we have $\eta_{e/h}(x, y) = x \in A \times B$. Because only the related points come into play, we will use the abbreviation $\eta$ for both $\eta_{e,h}$ and $\eta_{h,e}$.

We also need to modify the definition of $e/h$-convergence of bifunctions of biv $(X^0, Y^0)$ to have the corresponding Definition 6 below. For $\phi$ in biv $(X, Y)$, recall that its domain is defined by $\text{dom}\phi = \text{dom} \phi^e \cap \text{dom} \phi^h$.

**Definition 6** (epi/hypo-convergence, biv) Let $\phi$ and $\phi^k$ be in biv $(X^0, Y^0)$. We say that $\phi^k$ $e/h$-converge to $\phi$ if

(a) $\forall x \in X, y \in Y \ni \phi^k(x, y) \to \phi(x, y)$;

(b) $\forall x \in X \ni \phi^k \to \phi$.

**Remark 2** The definition of $e/h$-convergence\textsuperscript{8} requires that $(a)$ and $(b)$ are fulfilled for all $y \in Y$ and $x \in X$ instead of $y \in \text{dom} \phi$ and $x \in \text{dom} \phi$. The following example shows that $\{ \phi^k \}$ $e/h$-converge in the sense of Definition 6, but not $e/h$-converge following\textsuperscript{3}. Assume that $X = Y = \mathbb{R}$ and $\phi^k$ are defined by

$$
\phi^k(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in [1, 1]^2 \\
- & \text{if } x \notin [1, 0], y \notin [0, 1].
\end{cases}
$$

Then, $\text{dom} \phi = (0, 1) \times (0, 1)$ and $\phi^k \to \phi$, following our definition. However, $\phi^k$ do not satisfy $(a)$ with $\eta_{e,h} \phi$ replaced by $\forall y \in Y$. Indeed, for $y = 1$ and $k \to \infty$, $\phi^k \to 0$. There do not exist $k \to \infty$ with $\text{lik} \phi^k(\cdot, 1) \to 0$.

**CHARACTERIZATIONS OF VARIATIONAL CONVERGENCE OF BIFUNCTIONS**

**Characterizations by epi- and hypo-convergence of related unifunctions**

**Proposition 1** (characterizations of $e/h$-convergence by slices)

(i) Condition (a) of $e/h$-convergence $\iff \forall x \in A^k \to x$, $\text{hlim} \phi^k(x^k, \cdot) \geq \phi(x, \cdot)$ with $\phi^k(x^k, \cdot)$ defined on $A^k$ and $\phi(x, \cdot)$ on $A$, and $\forall x \in A^k \to x \notin A$. $\exists y \in B^k$, $\phi^k(x^k, y^k) \to \infty \iff \forall x \in A^k \to x$, one has (b) of $\phi^k(x^k, \cdot) \to \phi(x, \cdot)$ with the domains defined as above, and $\forall x \in A^k \to x \notin A$. $\exists y \in B^k$, $\phi^k(x^k, y^k) \to +\infty$.

(ii) Condition (b) of $e/h$-convergence $\iff \forall y \in B^k \to y$, $\text{elsub} \phi^k(\cdot, y^k) \geq \phi(\cdot, y)$ with $\phi^k(\cdot, y^k)$ defined on $A^k$ and $\phi(\cdot, y)$ on $A$, and $\forall y \in B^k \to y \notin B$. $\exists x \in A^k$, $\phi^k(x^k, y^k) \to -\infty \iff \forall y \in B^k \to y$. one has (a) of $\phi^k(\cdot, y^k) \to \phi(\cdot, y)$ with the domains defined as above, and $\forall y \in B^k \to y \notin B$. $\exists x \in A^k$, $\phi^k(x^k, y^k) \to -\infty$.

**Proof** (i) We show only the first equivalence. Condition (a) of $e/h$-convergence is that for all $y \in B$ and $x^k \in A^k \to x$, there exist $y^k \in B^k \to y$ such that $\text{lik} \phi^k(x^k, y^k) \geq \phi(x, y)$ when $x \in A$ together with the upper hyp-limit defined in part II. Moreover, the infinity condition in (a) of $e/h$-convergence is just the rest in the second side of this first equivalence. Similarly, applying this condition (a) together with the definition of hypo-convergence, we obtain the second equivalence.

(ii) The arguments are similar to those in (i) due to the symmetric property of epi/hypo-convergence.
Proposition 2 (characterizations of minsup-lop convergence by slices)

(i) Condition (a) of minsup-lop convergence is the same (a) of epi/hypo-convergence and so we have the same two equivalences in Proposition 1
(ii) Condition (b) of minsup-lop convergence $\Leftrightarrow \forall x \in A, \exists y^k \in A^k \to x$ such that $hs\phi^k(x^k, x)$ and $\phi(x^k, x)$ defined on $B$ and $\lim_{y \to x} \phi(y, x) = \infty$ when $y \in B$.

Proof: We need to prove only (ii). Condition (b) of minsup-lop convergence is $\forall x \in A, \exists y^k \in A^k \to x$ such that $\lim_{y \to x} \phi(y, x) = \infty$ when $y \in B$.

Corollary 1: Let $\phi^k \xrightarrow{\text{msi}} \phi$.

(i) There exist $y^k \in B^k \to y \in B$ such that $\phi(x^k, x) \leq \phi^k(x^k, y^k)$ for all $x^k \in A^k \to x$ and all $y^k \in B^k \to y$. Hence, $\forall x \in A, \exists y^k \in A^k \to x$ such that $\lim_{y \to x} \phi(y, x) = \infty$ when $y \in B$.

(ii) If $\phi^k$ do not depend on $y$ (or, resp), then $\phi(x^k, x) \leq \phi(\cdot, \cdot)$.

Remark 3: (i) The characterization of maxinf-lop convergence in terms of slices clearly states as follows: for $\phi^k \xrightarrow{\text{msi}} \phi$,

(a) There exist $y^k \in B^k \to y \in B$ such that $\phi(x^k, x) \leq \phi^k(x^k, y^k)$ for all $x^k \in A^k \to x$ and all $y^k \in B^k \to y$.

(b) When $\phi^k$ do not depend on $y$ (or, resp), then $\phi(\cdot, \cdot) \leq \phi^k(\cdot, \cdot)$.

(ii) We also have an assertion corresponding to (i) in Corollary 1, but with attention on changes on the roles of $(x, y)$ and of epi- and hypo-convergence. However, there is no direct connection between the lop-convergence of $\phi^k$ to $\phi$ and $\phi(\cdot, y)$ for all $y$ and $\phi^k(\cdot, \cdot)$ to $\phi^k(\cdot, \cdot)$ for all $x$. If $\phi^k$ depend only on one component, the equivalence (ii) becomes a sufficient condition for the lop-convergence of $\phi$.

Next, we discuss a relation between lop-convergence of bifunctions and convergence of other unifunctions being its sup- and inf-projections defined as follows.

The sup-projection (inf-projection, resp) of $\phi$ is a function $\phi^k(x, y) \leq \phi^k(x^k, y^k)$ defined on $B$.

To establish this relation, we need the following concept of ancillary tight convergence.

Definition 7 (ancillary tight e/h- and lop-convergence)

(i) (e/ancillary tight e/h-convergence) $\phi^k$ are called e/ancillary e/h-convergent to $\phi$ if (a) of e/h-convergence and the following condition are satisfied:

$$\forall y \in B, \forall x \in A \exists y^k \in A^k \to y, \exists x^k \in A^k \to x, y^k \in B^k \to y, \exists x^k \in A^k \to x, \lim_{y \to x} \phi(y, x) = \infty$$

(ii) (y-ancillary tight e/h-convergence) $\phi^k$ are called y-ancillary e/h-convergent to $\phi$ if (b) of e/h-convergence is fulfilled together with the condition

$$\forall x \in A, \forall y \in B, \forall x^k \in A^k, \exists y^k \in B^k \to y, \lim_{y \to x} \phi(y, x) = \infty$$

Proof: We prove the equivalence conclusion: $\phi^k \overset{\text{e/h}}{\xrightarrow{\text{msi}}} \phi$.

$$\phi^k \overset{\text{msi}}{\xrightarrow{\text{e/h}}} \phi \iff \forall x \in A, \exists y^k \in A^k \to x$$
any \((x, a)\) in the left-hand side, which means there are\(k\) such that \((x^k, a^k) \in \text{epi} \xi^h\). We claim that \(x \in \text{dom} \xi\). Clearly, \(x \in A\), because otherwise condition (a) of minsupp-lop convergence yields \(y^k \in B^h\) such that \(\phi^h(y^k, x^k) \rightarrow +\infty\), which contradicts the fact that \(a^k \geq \text{sup}\phi^h(y^k, \cdot)\) and \(a^k \rightarrow a\). Suppose \(\zeta(x) = +\infty\). Then, for any \(y > a\), there exists \(y \in B\) with \(\phi(x, y) > \gamma\). In view of the above (a), there exist \(\xi(x) \in B^h\) such that \(\lim \alpha_i \xi_i \xi(x) \geq \phi(x, y)\). Hence, \(\xi \leq \lim \alpha_i \xi_i \xi(x) \geq \phi(x, y)\). This contradiction shows that \(x \in \text{dom} \xi\).

For any \(\varepsilon > 0\), there exists \(y \in B\) with \(\phi(x, y) > \varepsilon\). The aforementioned (a) again gives \(\lim y^k \in B^h\) such that \(\lim \alpha_i \xi_i \xi(x) \geq \phi(x, y)x^k \geq \xi(x) - \varepsilon\). Therefore, as \(\varepsilon\) is arbitrary, \(\xi \leq \lim \alpha_i \xi_i \xi(x) \geq \xi(x)\). So, \(\xi \in \text{epi} \phi^h\). To prove the inclusion \(\text{epi} \phi^h \subset \text{Li} \text{epi} \phi^h\), observe that \(\phi^h\) minsupp-lop convergence x-ancillary tightly to \(\phi\) implies that for any \(x \in A\), there exists \(x^k \in A^h\) ensuring the tight hyp-convergence of \(\phi^h\) to \(\phi(x, \cdot)\). Using the counterpart of Theorem 2 (i) for hyp-convergence, one has \(\xi^h(x^k) \rightarrow \xi(x)\).

For any \((a, a) \in \text{epi} \xi\), i.e., \(\alpha \geq \xi(x)\), clearly, \(\xi^h(a^k + a - \xi(x)) \in \text{epi} \xi^h \rightarrow (x, a)\). Therefore, \(\text{epi} \phi^h \subset \text{Li} \text{epi} \phi^h\). We omit the statements about inf-projections corresponding to the above propositions.

**Characterizations by proper bifunctions**

**Theorem 3** For \(\phi^h\) in \(f\text{-biv}(X', Y)\), \(\phi^h\) \(e\text{-h}\)-converges to \(\phi\) if and only if \(\eta \phi^h\) \(e\text{-h}\)-converges to \(\eta \phi\).

**Proof** We use the explicit formula of \(\eta \phi^h\), but denotes it by \(\eta\) for simplicity.

(a) of Definition 6 \(\Rightarrow\) (a) of Definition 4. Suppose \(y \in B\) dom\(\phi\) and \(x^k \in A^h\). Then, (a) yields \(x^k \in A^h\) such that \(\lim \eta(a^k) = \inf\{\eta \phi^h(x^k, y^k)\} \geq \eta(x, y)\). (1)

Consider first \(x \in A\). Suppose the existence of \(y^k \in B^h\). Then, \(\eta(a^k) = \eta \phi^h(x^k, y^k) \rightarrow +\infty\), contradicting (1). Hence, \(x^k \in B^h\) and (1) means that \(\lim \eta(a^k) = \eta \phi^h(x^k, y^k) \geq \phi(x, y)\). Now let \(x \in A\). Then, \(\eta \phi^h(x^k, y^k) \rightarrow +\infty\) implies that \(\eta \phi^h(x^k, y^k) \rightarrow +\infty\). Again \(y^k \in B^h\) would give \(\eta \phi^h(x^k, y^k) \rightarrow +\infty\), contradicting (1). Hence, \(x^k \in B^h\), so \(\eta \phi^h(x^k, y^k) \rightarrow +\infty\) as (a) requires.

(a) \(\Rightarrow\) (a'). Let \(y \in B\) dom\(\phi\) and \(x^k \rightarrow x \in X\). If \(x \in A\) and there exist \(x^k \in A^h\), then (a) gives \(y^k \in B^h\) such that \(\lim \eta(a^k) = \inf\{\eta \phi^h(x^k, y^k)\} \rightarrow \eta(x, y)\). (2) We build a sequence \(y^k\) by inserting, for \(k \in \mathbb{N}\), \(y^k \in B^h\) so that \(y^k \rightarrow y\). Then, for \(k \in \mathbb{N}\), \(\lim \eta(a^k) = \inf\{\eta \phi^h(x^k, y^k)\} \rightarrow +\infty\) and does not effect the liminf in (2). Hence, we have (1). In the case that \(x^k \in A^h\), we can take any \(y^k \in B^h\) to obtain \(\lim \eta(a^k) = \inf\{\eta \phi^h(x^k, y^k)\} \rightarrow +\infty\) and then get (1).

**Geometric characterizations**

To characterize \(e\text{-h}\) and \(e\text{-l}\) convergence of \(\{\phi^k\}_k\) to \(\phi\) in a geometric way, we introduce the following six limit bifunctions of \(\{\phi^k\}_k\) in connection with \(\phi\).

**Definition 8** (six limit bifunctions) Let \(\phi\) and \(\phi^k\) be in \(f\text{-biv}(X', Y)\) and \((x, y) \in X \times Y\).

(i) (lower lop-limit bifunctions)

\[ \text{llls} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{lls} \phi^k(x, y) \]

for \(y \in B\).

(ii) (upper lop-limit bifunctions)

\[ \text{lls} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{lls} \phi^k(x, y) \]

for \(x \in A\).

(iii) (lower and upper e-h limit bifunctions)

\[ \text{ehl} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{ehl} \phi^k(x, y) \]

if \(y \in B\), \(x \in A\).

\[ \text{ehl} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{ehl} \phi^k(x, y) \]

if \(x \in A\).

\[ \text{ehl} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{ehl} \phi^k(x, y) \]

if \(y \in B\), \(x \in A\).

\[ \text{ehl} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{ehl} \phi^k(x, y) \]

if \(x \in A\).

\[ \text{ehl} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{ehl} \phi^k(x, y) \]

if \(y \in B\), \(x \in A\).

\[ \text{ehl} \phi^k(x, y) := \inf \left\{ \frac{\phi^k(x, y)}{i} \text{sup} \right\} \text{ehl} \phi^k(x, y) \]

if \(x \in A\).
Remark 4 Straightforwardly from the definitions one always has
\[ \text{lls}\phi^k(x,y) \leq \phi^k(x,y) \leq \text{ehls}\phi^k(x,y) \]
for any \( x, y \in X \times Y \).

Theorem 4 (formulas of the six limit bifunctions) Let \( \phi \) and \( \phi^k \) be in f/v-biv \((X \times Y)\) and \((x, y) \in X \times X \).
(i) (lower sup-limit bifunctions)
\[ \text{lls}\phi^k(x,y) := \max \{ x \in A : y \in B \} \]
for any \( x \in A, y \in B \).

(ii) (upper sup-limit bifunctions)
\[ \phi^k(x,y) := \sup \{ x \in A : y \in B \} \]
for any \( x \in A, y \in B \).

(iii) (lower and upper \( e/h \)-limit bifunctions)
\[ \text{ehls}\phi^k(x,y) := \max \{ x \in A : y \in B \} \]
\[ \phi^k(x,y) := \sup \{ x \in A : y \in B \} \]
for any \( x \in A, y \in B \).

Proof For reasons of simplicity, we discuss only (i).
To prove the first formula, denote \( u := \text{lls}\phi^k(x,y) \).
If \( u = +\infty \), any \( \{ x^* \in A : y \rightarrow \} \) is a minimizer for the considered expression. If \( u = -\infty \), for any \( y_j \downarrow 0 \), there exist \( x_j \in A \) such that for all \( y_j \rightarrow +\infty \), \( \text{lls}\phi^k(x_j,y_j) \downarrow -\infty \).

Hence, the sequence \( \{ x_j \} \) is a minimizer. If \( u = -\infty \), for any \( y_j \rightarrow -\infty \), one finds \( x_j \in A \rightarrow x \) such that for all \( x_j \in A \), \( y_j \rightarrow -\infty \), \( \text{lls}\phi^k(x_j,y_j) \downarrow -\infty \).
Similarly as above, with that corollary we see that the obtained \( \{ x_j \} \) is a minimizer.

Consider the second formula. Fix a sequence \( \{ x_k \} \rightarrow A \) and denote \( s := \sup \{ x_k : y \in B \} \).

We check the achievement of the involved supremum. If \( s = -\infty \), any \( \{ y_k \in B : x \rightarrow 0 \} \) gives the maximum.
If \( -\infty < s < +\infty \), for any \( y_j \rightarrow 0 \), there exist \( x_k \in B \) with \( \text{lls}\phi^k(x_k,y_k) \geq s - e_j \) for each \( j \in N \).
Taking \( l_k \) we have \( l_k \text{lls}\phi^k(x_k,y_k) \geq s \).

In view of Lemma A.1, we have \( k \rightarrow j(k) \) for each \( k \) such that \( l_k \text{lls}\phi^k(x_k,y_k) \geq s \). Of course the sequence \( \{ x_k \} \)
gives the aforementioned maximum. By the arbitrariness of \( \{ x_k \} \), we obtain the required equality. If \( s = +\infty \), then for any \( \theta^j \rightarrow +\infty \), one finds \( y_j \in B \) such that \( l_k \text{lls}\phi^k(x_k,y_k) \geq \theta^j \) for any \( j \). Taking \( l_j \) we arrive at \( l_j l_k \text{lls}\phi^k(x_k,y_k) \rightarrow +\infty \). Again by Lemma A.1, one gets \( j(k) \) for each \( k \) such that \( l_k \text{lls}\phi^k(x_k,y_k) \rightarrow +\infty \).
Hence, \( \{ x_k \} \) is a maximizer for the supersup expression in the definition of \( s \).

Theorem 5 (characterization of e/h-convergence) Let \( \phi \) and \( \phi^k \) be in f/v-biv \((X \times Y)\). Then, \( \phi^k \) converges to \( \phi \) if and only if the following two assertions hold:
(a) \( \phi^k(x,y) \leq \phi(x,y) \) whenever \( x \in A \) and \( y \in B \), and \( \text{lls}\phi^k(x) = +\infty \) when \( x \notin A \);
(b) \( \phi(x,y) \leq \phi^k(x,y) \) whenever \( y \in B \) and \( x \in A \), and \( \text{lls}\phi^k(x) = +\infty \) when \( y \notin B \).

Proof From Definition 4 and this theorem, it is obvious that (a) and (b) of e/h-convergence coincide with
(a) and (b), resp., of Theorem 5.

Observe that for \( \phi^k \) f/v-biv \((X \times Y)\), which of ehls\phi^k(x,y) and ehls\phi^k(x,y) is smaller changes from point to point in general. Theorem 5 stipulates that if ehls\phi^k(x,y)ehls\phi^k(x,y) for all \( (x,y) \in A \times B \), any \( \phi \) satisfying the inequality ehls\phi^k(x,y)ehls\phi^k(x,y) on \( A \times B \) and fulfills also the infinity items (for \( x \) outside \( A \) or \( y \) outside \( B \)) is an e/h-limit. Thus, in general the limits of a given e/h-convergent sequence, if exist, form an e/h-equivalence class of bifunctions.

Theorem 6 (characterizations of lop-convergence) Let \( \phi \) and \( \phi^k \) be in f/v-biv \((X \times Y)\).
(i) \( \phi^k \) min-sup-lim converge to \( \phi \) if and only if for \( x \in A \), \( \text{lls}\phi^k(x,y) \leq \phi(x,y) \) whenever \( y \in B \), and \( \text{lls}\phi^k(x) = +\infty \) whenever \( x \notin A \). For any \( x \notin A \), \( \text{lls}\phi^k(x) = +\infty \).

Hence, we have in fact the equalities for \( (x,y) \in A \times B \).

(ii) \( \phi^k \) max-inf-lim converge to \( \phi \) if and only if for \( y \in B \), \( \text{ehls}\phi^k(x,y) \leq \phi(x,y) \) whenever \( x \in A \), and \( \text{lls}\phi^k(x) = +\infty \) whenever \( x \notin A \) whenever \( y \notin B \). For any \( y \notin B \), \( \text{lls}\phi^k(x) = -\infty \).

Hence, we have in fact the equalities for \( (y,x) \in A \times B \).

Proof (i) Following the second formula in Theorem 4 (ii) and (iii)), condition (a) of min-sup-lim convergence means that \( (x,y) \leq \text{ehls}\phi^k(x,y) \) whenever \( x \in A, y \in B \), and \( \text{lls}\phi^k(x) = +\infty \) whenever \( x \notin A \), which is a part of assertion (i).

For \( x \in A \) and \( y \rightarrow B \), take any \( e \rightarrow 0 \) and \( y' \nearrow +\infty \).
We define
\[ \theta^j(x,y) := \begin{cases} \phi(x,y) + e_j \quad \text{if } y \rightarrow B, \\ -y' \quad \text{if } y \rightarrow B. \end{cases} \]
The remaining part of (i) means that, for \( x \in A \) and \( y' \rightarrow Y \),
\[ \text{lls}\phi^k(x,y) \begin{cases} \phi(x,y) \quad \text{if } y \rightarrow B, \\ -\infty \quad \text{if } y \rightarrow B. \end{cases} \]
This is equivalent to the statement: for each \( j \), there exist \( x^k_j \in A^k \rightarrow x \), for all \( y_j^k \in B^k \rightarrow y \), \( l\bar{s}_k \phi^k(x^k_j, y^k_j) \leq \bar{\theta}^k(x,y) \). Taking \( l\bar{s}_j \) of both sides yields

\[
\begin{align*}
    l\bar{s}_i l\bar{s}_k \phi^k(x^k_j, y^k_j) &= \phi(x,y) \quad \text{if} \quad y \bar{\in} B, \\
    &\leq \phi(x,y) \quad \text{if} \quad y \bar{\not\in} B.
\end{align*}
\]

By virtue of Corollary A3 in \(^8\), one has \( j(k) \) for each \( k \) such that

\[
\begin{align*}
    l\bar{s}_i l\bar{s}_k \phi^k(x^k_j(k), y^k_j(k)) &= \phi(x,y) \quad \text{if} \quad y \bar{\in} B, \\
    &\leq \phi(x,y) \quad \text{if} \quad y \bar{\not\in} B.
\end{align*}
\]

Therefore, the sequence \( \{x^k\}_k := \{x^k_j(k)\}_k \) is a minimizer for the expression of \( l\bar{s}_i l\bar{s}_k \phi^k(x,y) \), i.e., we obtain the remaining part of assertion (i).

(ii) The proof is similar to (i), but applying Theorem 4 with the first formula of (iii), both the formulas of (i), and Lemma A1 \(^8\) instead of the formulas of Theorem 4 applied in (i) and Corollary A37 , resp.

By Theorem 6, we see that, if exists, each of the minsup-lop limit and maxinf-lop limit is unique. Furthermore, both minsup-lop and maxinf-lop limits exist at the same time if and only if, for \( (x,y) \in A \times B \), \( l\bar{s} \phi^k = l\bar{s} \bar{\phi} \phi^k = \phi = l\bar{s} \bar{\phi} \phi \phi^k = l\bar{s} \phi^k \) together with the four infinity conditions stated in Theorem 6.

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CONFLICT OF INTERESTS

The author declares that there is no conflict of interest in publishing this paper.

AUTHORS’ CONTRIBUTIONS

This study is finished by one author.

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