Optimality conditions for set-valued optimization problems via scalarization function

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ABSTRACT
One of the most important and popular topics in optimization problems is to find its optimal solutions, especially Pareto optimal points, a well-known solution introduced in multi-objective optimization. This topic is one of the oldest challenges in many issues related to science, engineering and other fields. Many important practical-problems in science and engineering can be expressed in terms of multi-objective/ set-valued optimization problems in order to achieve the proper results/ properties. To find the Pareto solutions, a corresponding scalarization problem has been established and studied. The relationships between the primal problem and its scalarization one should be investigated for finding optimal solutions. It can be shown that, under some suitable conditions, the solutions of the corresponding scalarization problem have uniform spread and have a close relationship to Pareto optimal solutions for the primal one. Scalarization has played an essential role in studying not only numerical methods but also duality theory. It can be usefully applied to get relationships/important results between other fields, for example optimization, convex analysis and functional analysis. In scalarization, we usually use a kind of scalarized-functions. One of the first and the most popular scalarized-functions used in scalarization method is the Gerstewitz function. In the paper, we mention some problems in set-valued optimization. Then, we propose an application of the Gerstewitz function to these problems. In detail, we establish some optimality conditions for Pareto/ weak solutions of unconstrained/ constrained set-valued optimization problems by using the Gerstewitz function. The study includes the consideration of problems in theoretical approach. Some examples are given to illustrate the obtained results.

Key words: Pareto efficient solution, weak efficient solution, set-valued optimization, Gerstewitz function, optimality condition

INTRODUCTION
Scalarization has an essential role in studying numerical methods and duality theory. It can be applied to get relationships between other fields, such as: optimization, convex analysis and functional analysis. Solutions of vector optimization problems can be characterized by those of corresponding scalarized optimization problems.

In scalarization, the Gerstewitz function plays an important role. Its main properties were studied in some papers.[5-7] In the paper, we have proposed optimality conditions for set-valued optimization problems using the Gerstewitz function. These results have contributed to applications of the Gerstewitz function in optimization.

RELIMINARIES
In the paper, Let X and Y be normed spaces, K be a pointed, closed, convex cone with nonempty interior in Y. For A be a nonempty subset in Y, we denote int(A), cl(A) and cone(A) for the interior, the closure of A and the cone generalized by A, respectively. Let F := X → 2Y be a set-valued map from X to Y, the domain, the image and the graph of F are defined by dom(F) := {x ∈ X|F(x) ≠ ∅}; im(F) := {y ∈ Y|y ∈ F(dom(F))}; gr(F) := {(x, y) ∈ X × Y|y ∈ F(x)}.

Definition 2.1. Let F := X → 2Y and (x₀, y₀) ∈ gr(F).

(i) A point (x₀, y₀) is called a Pareto efficient solution of F on X if (F(X) − y₀) ∩ (−K \ {0}) = ∅. The set of Pareto efficient solutions of F is denoted by Minₓ F(X).

(ii) A point (x₀, y₀) is called a weak efficient solution of F on X if (F(X) − y₀) ∩ (−int(K) = ∅. The set of weak efficient solutions of F is denoted by WMinₓ F(X).

Note that Minₓ F(X) is a subset of WMinₓ F(X). In general, the inverse conclusion is not true by the following example.

Example 2.2. Let X = R, Y = R², k = R₂⁺, F₁, F₂ : X → 2Y be defined by

Definition 2.3. Let \( X, Y, Z \) be optimization problems as follows.

\[
\begin{align*}
F_1(x) := \left\{ (y_1, y_2) \in Y : y_1 \geq y_2 \right\}; \\
F_2(x) := \left\{ (y_1, y_2) \in Y : y_1 \geq y_2, y_2 \leq 1 \right\} \cup \left\{ (y_1, y_2) \in Y : y_1 \geq 0, y_2 \geq 0 \right\}.
\end{align*}
\]

With \( F_1 \), one has \( \min_K F_1(X) \) and \( \max_K F_1(X) \) where \( (x, y_1, y_2) \in X \times Y \), \( y_1 = y_2 \), \( y_2 \leq 1 \), \( y_2 \leq 1 \), \( y_1 \geq 0 \), \( y_2 \geq 0 \).

With \( F_2 \), we get \( \min_K F_2(X) = \left\{ (x, y_1, y_2) \in X \times Y : y_1 = y_2, y_2 \leq 1 \right\} \cup \left\{ (x, y_1, y_2) \in X \times Y : y_1 = 0, y_2 \geq 0 \right\} \).

Thus, \( \min_K F_2(X) \) is a subset of \( \max_K F_2(X) \).

**Definition 2.3.** With \( e \in \text{int} (K) \), the Gerstewitz function \( h^e_K(y) := \inf \{ t \in \text{int}(Y) : y \geq te \} \).

If \( K \) is a pointed, closed, convex cone with nonempty interior then \( h^e_K \) is finite and we get

\[
h^e_K(y) = \sup \{ h(y) : h \in K, h(e) = 1 \}, \quad \forall y \in Y,
\]

where \( K := \{ k \in \text{int}(Y) : |\{ k \} \geq 0, \forall k \in K \} \). It is a convex and continuous function on \( Y \). We recall some properties of the Gerstewitz function as follows.

**Proposition 2.4.** With \( e \in \text{int} (K) \), we have

\[
\begin{align*}
& (i) -\infty \leq e - K = \{ y \in Y : h^e_K(y) \geq 0 \}; \\
& (ii) -\infty = e - K = \{ y \in Y : h^e_K(y) > 0 \}; \\
& (iii) -\infty = \{ (-\infty, 0) e - K \} = \{ y \in Y : h^e_K(y) = 0 \}; \\
& (iv) \{ (-\infty, 0) e - K \} = \{ y \in Y : h^e_K(y) \geq 0 \}; \\
& (v) \{ (-\infty, 0) e - K \} = \{ y \in Y : h^e_K(y) > 0 \}; \\
& (vi) \forall \alpha > 0, h^e_K(\alpha y) = \alpha h^e_K(y). \\
\end{align*}
\]

For applications of the Gerstewitz function, the reader is referred to the references.\(^6\)-\(^9\).

**Optimality Conditions**

Let \( X, Y, Z \) be normed spaces, \( K \) and \( D \) be pointed, closed, convex cones with nonempty interior in \( Y \) and \( Z \), respectively, \( F : X \rightarrow 2^Y \), \( G : X \rightarrow 2^Z \). We consider two optimization problems as follows.

\[
(P_1) = \left\{ \min_K F(x), \quad s.t.: x \in X \right\};
\]

\[
(P_2) = \left\{ \min_K F(x), \quad s.t.: x \in X, G(x) \cap (-D) \neq \emptyset \right\}.
\]

Let \( S_i \) be feasible sets of \( (P_i), i=1,2 \), then \( S_1 = X \) and \( S_2 = \{ x \in X : G(x) \cap (-D) \neq \emptyset \} \).

A point \( (x_0, y_0) \) is a weak efficient solution of \( (P_1) \) if and only if \( x_0 \in S_1 \) and \( F(S_1 \setminus y_0) \cap (-K) = \emptyset \).

**Theorem 3.1.** Let \( (x_0, y_0) \) be a weak efficient solution of \( (P_1) \) if and only if \( \exists \gamma \in F(S_1) \) such that \( h^e_K(y - y_0) \geq 0 \).

Proof. Let \( \gamma \in F(S_1) \) such that \( h^e_K(y - y_0) \geq 0, \forall y \in F(S_1) \).

**Theorem 3.2.** Let \( (x_0, y_0) \) be a weak efficient solution of \( (P_1) \) if and only if \( \exists \gamma \in F(S_1) \) such that \( h^e_K(y - y_0) \geq 0, \forall y \in F(S_1) \).

Proof. Let \( \gamma \in F(S_1) \) such that \( h^e_K(y - y_0) \geq 0, \forall y \in F(S_1) \).

**Theorem 3.3.** Let \( (x_0, y_0) \) be a weak efficient solution of \( (P_1) \) if and only if \( \exists \gamma \in F(S_1) \) such that \( h^e_K(y - y_0) \geq 0, \forall y \in F(S_1) \).

Proof. Let \( \gamma \in F(S_1) \) such that \( h^e_K(y - y_0) \geq 0, \forall y \in F(S_1) \).

Hence, \( (x_0, y_0) \) is a weak efficient solution of \( (P_1) \).
Suppose that \((x_0, y_0)\) is not a weak efficient solution of \((P_2)\), there exist \(x \in \tilde{X}, z \in \tilde{G}(x) \cap (-D), y \in \tilde{F}(x)\) such that \(y - y_0 \in -\text{int} K\). Therefore, 
\[ y + \tilde{T}(z) - y_0 \in -\text{int} K - K = -\text{int} K, \]
which contradicts to (3). Hence, \((x_0, y_0)\) is a weak efficient solution of \((P_2)\).

“Only if”: Suppose that (2) does not hold, i.e., for all \(e \in \text{int}(K)\), \(T \in \Pi\) one has \(x \in X\) such that
\[ h_e^e(L(x, T) - y_0) < 0. \]
By Proposition 2.4(ii), we obtain
\[ L(x, T) - y_0 \subseteq (-\infty, 0) e - K \subseteq -\text{int} K. \]
It means that \(F(x) + T(G(x)) - y_0 \subseteq -\text{int} K\). For \(x\) with \(G(x) \cap (-D) \neq \emptyset\), there is \(z \in G(x) \cap (-D), y \in \tilde{F}(x)\) such that \(y - y_0 \in -\text{int} K\), which is a contradiction. Hence, (2) is fulfilled. ■

**Theorem 3.4.** Let \((x_0, y_0) \in gr(F)\) with \(x_0 \in S_2\) and \(z_0 \in G(x_0) \cap (-D)\).

(i) If there exist \(e \in \text{int}(K)\), there exist \(e \in \text{int}(K)\) with (2) holds \(\forall x \in X, y \in L(x, T)\).

(ii) If there exist \(e \in \text{int}(K)\), \(T \in \Pi\) such that (2) holds with strict inequality \(\forall x \in X, y \in L(x, T)\), then \((x_0, y_0)\) is a Pareto efficient solution of \((P_2)\).

**Proof.** (i) Since \((x_0, y_0)\) is a Pareto efficient solution of \((P_2)\), \((x_0, y_0)\) is a weak efficient solution of \((P_2)\). By Theorem 3.3, we are done.

(ii) Since (2) holds with strict inequality, by Proposition 2.4(v), one gets \(L(x, T) - y_0 \subseteq ((-\infty, 0) e - K) = Y \setminus (-K)\), or (4).

\[ F(x) + T(G(x)) - y_0 \subseteq Y \setminus (-K). \]  

Suppose that \((x_0, y_0)\) is not a Pareto efficient solution of \((P_2)\), there exist \(x \in \tilde{X}, z \in \tilde{G}(x) \cap (-D), y \in \tilde{F}(x)\) such that \(y - y_0 \in -K \setminus \{0\}\). Therefore, 
\[ y + \tilde{T}(z) - y_0 \in -K \setminus \{0\} - K \subseteq -K, \]
which contradicts to (4). Hence, \((x_0, y_0)\) is a Pareto efficient solution of \((P_2)\). ■

To illustrate Theorem 3.3, we consider the following example.

**Example 3.5.** Let \(X = Y = Z = \mathbb{R}, K = D = \mathbb{R}_+, F : X \rightarrow 2^Y, G : X \rightarrow 2^D\) be defined by
\[ F(x) := \{ y \in Y | 0 \leq y \leq x^2 \}; G(x) := \{ z \in Z | 0 \leq z \leq |x| \}. \]
With \((P_2)\), one has \(\{ x \in X | G(x) \cap (-D) \neq \emptyset \} = \mathbb{R}\).
It is easy to see that \((x_0, y_0) = (0, 0)\) is a weak efficient solution of \((P_2)\). We now check that condition (2) holds. In fact, \(e = 1, T(x) = x\) is a linear operator satisfying \(T(D) K\), then
\[ L(x, T) := F(x) + T(G(x)) = [0, x^2] + [0, |x|] = [0, x^2 + |x|]. \]
For all \(y \in L(x, T)\), \(x \in X\), one gets 
\[ h_e^e(y - y_0) := \inf \{ t \in \mathbb{R} | y \in t e - K \} = \inf \{ t \in \mathbb{R} | 0, x^2 + |x| \in t - K \} = x^2 + |x| \geq 0. \]

**CONCLUSIONS**

In the paper, we first recall the Gerstewitz scalar function and its basic properties. Via this function, optimality conditions for some kinds of optimization problems are established concerning Pareto efficient/weak efficient solutions.

In set-valued optimization, we have several kinds of solutions, such as: Geoffrion efficient solution \(^{10}\), Borwein efficient solution \(^{11}\), Benson efficient solution \(^{12}\). They have been studied in some recent results \(^{13,14}\). Thus, for the possible development, we think that giving optimality conditions for the above-mentioned solutions may be a promising approach.

**AUTHOR’S CONTRIBUTIONS**

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

**CONFLICT OF INTEREST**

We declare that there is no conflict of whatsoever involved in publishing this research.

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