Science & Technology Development Journal – Engineering and Technology, 3(SI3):SI58-SI62

Open Access Full Text Article

Research Article

Existence conditions for symmetric strong vector quasi-equilibrium problems

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History

- Received: 6-12-2019
- Accepted: 18-12-2020
- Published: 31-12-2020

DOI :10.32508/stdjet.v3iSI3.638

ABSTRACT

In the following paper, the symmetric strong vector quasi-equilibrium problems will be studied thoroughly. Afterward, the existence conditions of solution sets for these problems has been established. The results which are presented in this paper improve and extend the main results mentioned in the literature. Our results can be illustrated by some interesting examples.

In 1994, Noor and Oettli introduced the following the symmetric scalar quasi-equilibrium problem. This problem is one of the generalization of the symmetric scalar quasi-equilibrium problem which is presented by Noor and Oettli. Since then, the symmetric vector quasi-equilibrium problem has been investigated by a huge number of authors in different ways.

The research works mentioned above are one of our motivation to improve and extend the problem. So, in this paper, we will introduce the vector quasi-equilibrium problems. Afterward, some existence conditions of solution sets for these problems will be established. The symmetric vector quasi-equilibrium problems consist of many optimization - related models namely symmetric vector quasi-variational inequality problems, fixed point problems, coincidence-point problems and complementarity problems, etc. In recent years, a lot of results for existence of solutions for symmetric vector quasi-equilibrium problems, vector quasi-equilibrium problems, vector quasi-variational inequality problems and optimization problems have been established by many authors in different ways.

We will present our work in the following steps. In the first section of our paper, we will introduce the model of symmetric vector quasi-equilibrium problems. In the following section, we recall definitions, lemmas which can be used for the main results. In the last section, we will establish some conditions for existence and closedness of the solutions set by applying fixed-point theorem for symmetric vector quasi-equilibrium problems.

The results presented in this paper improve and extend the main results in the literature. Some examples are given to illustrate our results. Hence our results, Theorem 3.1 and Theorem 3.6 have significant improvements.

Key words: Symmetric generalized quasi-equilibrium problems, Kakutani-Fan-Glicksberg fixed-point theorem, Closedness

INTRODUCTION

In 1994, Noor and Oettli[1] presented the following the symmetric scalar quasi-equilibrium problem which consists of finding $(\bar{x},\bar{y}) \in A\times B$ such that $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$

and

$f(x,\bar{y}) \geq f(\bar{x},\bar{y}),\ for\ all\ x \in S(\bar{x},\bar{y}),$

$g(\bar{x},y) \geq g(\bar{x},\bar{y}),\ for\ all\ y \in T(\bar{x},\bar{y}),$

where X, Y are real locally convex Hausdorff topological vector spaces and $A \subset X, B \subset Y$ are non-empty sets, $S : A\times B \rightrightarrows A,\ T : A\times B \rightrightarrows B$ are set-valued mappings and $f,g : A\times B \to R$ are real functions.

In 2003, the symmetric vector quasi-equilibrium problem (in short, (SVQEP)) has been introduced and investigated by Fu[2]. Let X, Y and Z be real locally convex Hausdorff topological vector spaces, and let $A\subset X$, $B\subset Y$ be non-empty sets and $C\subset Z$ be a closed convex point cone with int $C\neq\emptyset$, where int C denotes the interior of C.

Let $S : A\times B \rightrightarrows A,\ T : A\times B \rightrightarrows B$ be set-valued mappings and f,g:$A\times B\longrightarrow Z$ be vector functions. Find $(\bar{x},\bar{y})\in A\times B$ such that $\bar{x}\in S(\bar{x},\bar{y}),\bar{y}\in T(\bar{x},\bar{y})$ and

$f(x,\bar{y}) - f(\bar{x},\bar{y}) \notin -\mathrm{int}C,$ for all $x \in S(\bar{x},\bar{y}),$

$g(\bar{x},y) - g(\bar{x},\bar{y}) \notin -\mathrm{int}C,$ for all $y \in T(\bar{x},\bar{y}).$

The problem is one of the generalization of the symmetric scalar quasi-equilibrium problem by Noor and Oettli[1]. Since then, many authors have studied the symmetric vector quasi-equilibrium problem in different ways, see[3,4] and the references therein.

In this paper, we will introduce the vector quasi-equilibrium problems. Afterward, we will point out some existence conditions of solution sets for these problems. Now, we move on to our problem setting.

Cite this article : Dai L X, Hung N V. **Existence conditions for symmetric strong vector quasi-equilibrium problems**. *Sci. Tech. Dev. J. – Engineering and Technology;* 3(SI3):SI58-SI62.

Let X,Y,Z be real locally convex Hausdorff topological vector spaces, $A\subset X,B\subset Y$ be non-empty compact subsets and $C\subset Z$ be a closed convex point cone with int $C \neq \varnothing$, where int C denotes the interior of C. Let $K: A\times B \rightrightarrows A$, $T: A\times B \rightrightarrows B$ be multi-functions and $f: A\times B\times A \to Z$, $g: B\times A\times B \to Z$ be vector functions. We will study the following symmetric vector quasi-equilibrium problems (in short, (SQVEP).

(SQVEP): Find $(\bar{x},\bar{y}) \in A\times B$ such that $\bar{x} \in S(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$ and

$$f(\bar{x},\bar{y},x^*) \in C, \forall x^* \in K(\bar{x},\bar{y}),$$
$$g(\bar{y},\bar{x},y^*) \in C, \forall y^* \in T(\bar{x},\bar{y}),$$

We denote that Ω is the solution set of (SQVEP).

Note that the symmetric vector quasi-equilibrium problems include many optimization - related models, for instance, symmetric vector quasi-variational inequality problems, vector quasi-equilibrium problems, variational inequality problems, Nash equilibria problems, fixed point problems, coincidence-point problems and complementarity problems, *etc*. In recent years, a lot of results for existence of solutions for symmetric vector quasi-equilibrium problems, vector quasi-equilibrium problems, vector quasi-variational inequality problems and optimization problems have been established by many authors in different ways. For more details on this topic, we refer the readers to [3–5] and the references therein.

Our paper is presented in the following way. Firstly, we introduce the model symmetric vector quasi-equilibrium problems. Secondly, we recall definitions, lemmas which can be used to proof the main theorem. Lastly, we establish some existence and closedness theorems by using fixed-point theorem for symmetric vector quasi-equilibrium problems.

PRELIMINARIES

In this section, we recall some basic definitions and some of their properties.

Definition 2.1[6] Let X,Y be two topological vector spaces, A be a nonempty subset of X and $F: A \rightrightarrows Y$ be a multi-function.

a) F is said to be lower *semi-continuous* (lsc) at $x_0 \in A$ if $F(x_0) \cap U \neq \varnothing$ for some open set $U\subset Y$ implies the existence of a neighborhood N of x_0 such that $F(x) \cap U \neq \varnothing, \forall x \in N$.

b) F is said to be lower *semi-continuous* in A if it is lower semicontinuous at all $x_0 \in A$.

c) F is said to be *upper semi-continuous* (usc) at $x_0 \in A$ if for each open set $U\supset F(x_0)$, there is a neighborhood N of x_0 such that

$$U \supset F(x),\ \forall x \in N$$

d) F is said to be *upper semi-continuous* in S if it is upper semicontinuous at all $x_0 \in A$.

e) F is said to be *continuous in* A if it is both lsc and usc in A.

f) F is said to be closed if

$$Graph(F) = \{(x,y) : x \in A, y \in F(x)\}$$

is a closed subset in $A\times Y$.

Lemma 2.1[7] *Let X,Y be two topological vector spaces, A be a nonempty convex subset of X and $F: A \rightrightarrows Y$ be a multifunction.*

a) If F is upper semicontinuous at $x_0 \in A$ with closed values, then F is closed at $x_0 \in A$.

b) If F is closed at $x_0 \in A$ and Y is compact, then F is upper semi-continuous at $x_0 \in A$.

c) If F has compact values, then F is usc at $x_0 \in A$ if and only if, for each net $\{x_\alpha\} \subseteq A$ which converges to $x_0 \in A$ and for each net $\{y_\alpha\} \subseteq F(x_\alpha)$, there are $y_0 \in F(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \longrightarrow y_0$.

Definition 2.2[8] Let X and Z be two topological vector spaces and $A\subseteq X$ be non-empty convex set, $C\subset Z$ is a nonempty closed convex cone. Suppose $f: A\longrightarrow Z$ be a vector function. f is called properly C-quasi-convex in A if and only if, for every $x_1, x_2 \in A$ and each $\lambda \in [0,1]$, we have

$$\begin{aligned} &\text{either } f(\lambda x_1 + (1-\lambda)x_2) \le f(x_1), \\ &\text{or} \quad f(\lambda x_1 + (1-\lambda)x_2) \le f(x_2). \end{aligned}$$

Definition 2.3[7] Let X and Z be two topological vector spaces and $A\subseteq X$ be nonempty convex set, $C\subset Z$ is a non-empty closed convex cone. A mapping $f: A\longrightarrow Z$ is called *C-continuous* at $x_0 \in A$ if, for any open neighborhood V of 0 in Z, there exists an open neighborhood U of x_0 in A such that

$$f(x) \in f(x_0) + V + C, \forall x \in U \cap A,$$

and *C-continuous* in A if it is *C -continuous* at every point of A.

Lemma 2.2[9] *Let A be a nonempty convex compact subset of Hausdorff topological vector space X and M be a subset of $A\times A$ such that*

a) for each at $x \in A, (x,x) \notin M$;

b) for each at $y \in A$, the set $\{x \in A: (x,x) \in M\}$ is open in A;

c) for each at $x \in A$, the set $\{y \in A: (x,x) \in M\}$ is convex or empty.

Then, there exists $x_0 \in A$ such that $(x_0,y) \notin M$ for all $y \in A$.

Lemma 2.3[10] *Let A be a non-empty compact convex subset of a locally convex Hausdorff vector topological space X. If $F: A \rightrightarrows A$ is upper semi-continuous and for any $x \in A$, F(x) is non-empty, convex and closed, then there exists $x^* \in A$ such that $x^* \in F(x^*)$.*

MAIN RESULTS

In this section, we establish some existence theorems of solution sets for (SQVEP).

Definition 3.1 Let X,Y,Z be topological vector spaces and $C\subset Z$ be a closed convex point cone with int $C \neq \varnothing$, where int C denotes the interior of C. Suppose $h{:}X,Y,X\longrightarrow Z$ be a vector function. h is said to be strongly C-quasiconvex (with respect to the first variable) in a set $A\subset X$, if for each $y\in Y$, $z\in X$ and $\forall x_1,x_2 \in A$, $\forall\lambda\in[0,1]$, $h(x_1,y,z)\in C$ and $h(x_2,y,z)\in C$. Then, it follows that

$$h(\lambda x_1+(1-\lambda)x_2,y,z)\in C.$$

Theorem 3.1 *Suppose that for the problem* (SQVEP) *that*

i) K and T are continuous in $A\times B$ with non-empty compact convex values;

ii) for all $(x,y)\in A\times B$, $f(x,y,x)\in C$ and $g(x,y,x)\in C$

iii) the set $\{(y,x^*)\in B\times A{:}\ f(.,y,x^*)\notin C\}$ is convex in A and the set

$\{(x,y^*)\in A\times B{:}\ g(.,x,y^*)\notin C\}$ is convex in B;

iv) for all $(y,x^*)\in B\times A$, $f(.,y,x^*)$ is strongly C-quasiconvex in A, and for all $(x,y^*)\in A\times B$, $g(.,x,y^*)$ is strongly C-quasi-convex in B;

v) the set

$\{(x,y,x^*)\in A\times B\times A{:}\ f(x,y,x^*)\in C\}$ is closed, and the set $\{(y,x,y^*)\in B\times A\times B{:}\ g(y,x,y^*)\in C\}$ is closed.

Then, the (SQVEP) has a solution, i.e., there exists $(\bar{x},\bar{y})\in A\times B$ such that $\bar{x}\in S(\bar{x},\bar{y})$, $\bar{y}\in T(\bar{x},\bar{y})$ and

$$f(\bar{x},\bar{y},x^*)\in C,\forall x^*\in K(\bar{x},\bar{y}),$$
$$g(\bar{y},\bar{x},y^*)\in C,\forall y^*\in T(\bar{x},\bar{y}),$$

Moreover, the solution set of the (SQVEP) is closed.

Proof. *For all $(x,y)\in A\times B$, define mappings: $\Pi_1 : A\times B\rightrightarrows A$ and $\Pi_2 : A\times B\rightrightarrows B$ by*

$$\Pi_1(x,y)=\left\{\begin{matrix} a\in K(x,y): f(a,y,x^*)\in C,\\ \forall x^*\in K(x,y)\end{matrix}\right\},$$

and

$$\Pi_2(x,y)=\left\{\begin{matrix} b\in T(x,y): g(b,y,x^*)\in C,\\ \forall y^*\in T(x,y)\end{matrix}\right\},$$

Firstly, we will show that $\Pi_1(x,y)$ and $\Pi_2(x,y)$ are non-empty.

Indeed, for all $(x,y)\in A\times B$, $K(x,y)$ is non-empty compact convex set. Setting

$$M=\left\{\begin{matrix}(a,x^*)\in K(x,y)\times K(x,y):\\ f(a,y,x^*)\notin 0\end{matrix}\right\}$$

(1) The condition (ii) we have, for any

$$a\in K(x,y),(a,a)\notin M$$

(2) The condition (iii) implies that for any $a\in K(x,y)$, $\{x^*\in K(x,y):(a,x^*)\in M\}$ is convex in $K(x,y)$.

3) The condition (v), we have for any $a\in K(x,y)$, $\{x^*\in K(x,y):(a,x^*)\in M\}$ is open in $K(x,y)$.

By Lemma 2.2, there exists $a\in K(x,y)$ such that $(a,x^*)\notin M$ for all $x^*\in K(x,y)$, i.e.,

$$f(a,y,x^*)\in C,\forall x^*\in K(x,y).$$

Thus $\Pi_1(x,y)=\varnothing$. Similarly, we also have $\Pi_2(x,y)=\varnothing$.

Secondly, we will prove that $\Pi_1(x,y)$ and $\Pi_2(x,y)$ are non-empty convex sets.

Let $a_1,a_2\in\Pi_1(x,y)$ and $\alpha\in[0,1]$ and let $a=\alpha a_1+(1-\alpha)a_2$. Since $a_1,a_2\in K(x,y)$ and $K(x,y)$ is a convex set, we have $a\in K(x,y)$. Thus, for $a_1,a_2\in\Pi_1(x,y)$, it follows that

$$f(a_1,y,x^*)\in C,\forall x^*\in K(x,y),$$

and

$$f(a_2,y,x^*)\in C,\forall x^*\in K(x,y).$$

By (iv), $f(.,y,x^*)$ is strongly *C-quasi-convex.*

$$f(ta_1+(1-t)a_2,y,x^*)\in C,\ \text{for all}\ t\in[0,1],$$

i.e. $a\in\Pi_1(x,y)$. Therefore, $\Pi_1(x,y)$ is convex.

Similarly, we have $\Pi_2(x,y)$ is convex.

Thirdly, we will proof that Π_1 and Π_2 are upper semi-continuous in $A\times B$ with non-empty compact values.

Firstly, we will show that Π_1 is upper semi-continuous in $A\times B$ with nonempty compact values. Indeed, since A is a compact set, by Lemma 2.1(ii), then we will show that Π_1 is a closed mapping. Indeed, let a net

$$\{(x_\alpha,y_\alpha):\alpha\in I\}\subset A\times B$$

such that

$$(x_\alpha,y_\alpha)\to(x,y)\in A\times B$$

and let $a_\alpha\in\Pi_1(x_\alpha,y_\alpha)$ such that $a_\alpha\longrightarrow a_0$. Now we need to show that $a_0\in\Pi_1(x,y)$. Since $a_\alpha\in K(x_\alpha,y_\alpha)$ and K is upper semi-continuous with non-empty compact values, hence K is closed, thus, we have $a_0\in K(x,y)$. Suppose the contrary $a_0\notin\Pi_1(x,y)$. Then, there exists $x_0^*\in K(x,y)$ such that

$$f(a_0,y,x_0^*)\notin C. \tag{3.1}$$

By the lower semi-continuity of K, there is a net $\{x_\alpha^*\}$ such that $x_\alpha^*\in K(x_\alpha,y_\alpha)$, $x_\alpha^*\to x_0^*$. Since $a_\alpha\in\Pi_1(x_\alpha,y_\alpha)$, we have

$$f(a_\alpha,y_\alpha,x_\alpha^*)\in C. \tag{3.2}$$

By the condition (v) and (3.2), we have

$$f\left(a_0, y, x_0^*\right) \in C. \tag{3.3}$$

This is the contradiction between (3.1) and (3.3). Thus $a_0 \in \Pi_1(x,y)$. Hence, Π_1 is upper semi-continuous in $A\times B$ with nonempty compact values. Similarly, we also have Π_2 is upper semi-continuous in $A\times B$ with non-empty compact values.

Next, we need to prove the solutions set $\Omega \neq \varnothing$.

Define the set-valued mappings

$\Phi_1 : A \times B \rightrightarrows A \times A$ and $\Phi_2 : A \times B \rightrightarrows B \times B$ by

$$\Phi_1(x,y) = (\Pi_1(x,y), K(x,y)), \forall (x,y) \in A, B$$

and

$$\Phi_2(x,y) = (\Pi_2(x,y), K(x,y)), \forall (x,y) \in A, B$$

Then Φ_1, Φ_2 are upper semi-continuous and $\forall (x,y) \in A \times B$, $\Phi_1(x,y)$ and $\Phi_2(x,y)$ are non-empty compact convex subsets of $A\times B$.

Define the set-valued mapping

$$H : (A \times B) \times (A \times B) \rightrightarrows (A \times A) \times (B \times B)$$

by

$$H((x,y),(x,y)) = (\Phi_1(x,y), \Phi_2(x,y))$$

Then H is also upper semi-continuous and $\forall (x,y) \in A \times B$, $H((x,y),(x,y))$ is a nonempty closed convex subset of $(A\times B)\times(A\times B)$.

By Lemma 2.3, there exists a point $((\widehat{x},\widehat{y}),(\widehat{x},\widehat{y})) \in (A \times B) \times (A \times B)$

such that

$$((\widehat{x},\widehat{y}),(\widehat{x},\widehat{y})) \in H((\widehat{x},\widehat{y}),(\widehat{x},\widehat{y}))$$

that is

$$((\widehat{x},\widehat{y}),(\widehat{x},\widehat{y})) \in \Phi_1(\widehat{x},\widehat{y}),\ (\widehat{x},\widehat{y}) \in \Phi_2(\widehat{x},\widehat{y})$$

which implies that $\widehat{x} \in \Pi_1(\widehat{x},\widehat{y})$, $\widehat{y} \in K(\widehat{x},\widehat{y})$ and $\widehat{x} \in \Pi_2(\widehat{x},\widehat{y})$, $\widehat{y} \in T(\widehat{x},\widehat{y})$. Hence,

$$\widehat{x} \in K(\widehat{x},\widehat{y}),\ \widehat{y} \in T(\widehat{x},\widehat{y})$$

and

$$f(\widehat{x},\widehat{y},x^*) \in C,\ \forall x^* \in K(\widehat{x},\widehat{y}),$$

and

$$g(\widehat{y},\widehat{x},y^*) \in C,\ \forall y^* \in T(\widehat{x},\widehat{y}),$$

i.e., (SQVEP) has a solution.

Finally, we prove that Ω is closed. Indeed, let a net $\{(x_\alpha,y_\alpha), \alpha \in I\} \subset \Omega :\ (x_\alpha,y_\alpha) \longrightarrow (x_0,y_0)$. We need to prove that $(x_0,y_0) \in \Omega$. Indeed, by the lower semi-continuity of K and T, for any

$$x_0 \in K(x_0,y_0),\ y_0 \in T(x_0,y_0),$$

there exist $x_\alpha \in K(x_\alpha,y_\alpha)$, $y_\alpha \in T(x_\alpha,y_\alpha)$ such that $x_\alpha \longrightarrow x_0$, $y_\alpha \longrightarrow y_0$. Since $(x_\alpha,y_\alpha) \in \Omega$, we have $x_\alpha \in K(x_\alpha,y_\alpha)$, $y_\alpha \in T(x_\alpha,y_\alpha)$ such that

$$f(x_\alpha,y_\alpha,x_\alpha^*) \in C, \forall x_\alpha^* \in K(x_\alpha,y_\alpha),$$

and

$$g(y_\alpha,x_\alpha,y_\alpha^*) \in C, \forall y_\alpha^* \in T(x_\alpha,y_\alpha).$$

Since K,T are upper semi-continuous in $A\times B$ with nonempty compact values. There exist $x_0^* \in K(x_0,y_0)$ and $y_0^* \in T(x_0,y_0)$ such that $x_\alpha^* \longrightarrow x_0^*$, $y_\alpha^* \longrightarrow y_0^*$ (taking subnets if necessary). By the condition (v) and

$$(x_\alpha,y_\alpha,x_\alpha^*,y_\alpha^*) \to (x_0,y_0,x_0^*,y_0^*),$$

we have

$$f(x_0,y_0,x_0^*) \in C, \forall x_0^* \in K(x_0,y_0),$$

and

$$g(y_0,x_0,y_0^*) \in C, \forall y_0^* \in T(x_0,y_0).$$

This means that $(x_0,y_0) \in \Omega$. Thus Ω is a closed set.

Remark 3.2 If $K(x,y) = K(x)$, $T(x,y) = T(x)$, $g(y,x,y^*) = f(x,y,x^*)$ with $x \in A$, $y \in A$, $x^* \in A$, $y^* \in B$. Then, (SQVEP) will be the strong vector quasi-equilibrium problem which was studied in [11]. Hou et al.[11] also obtained an existence result for strong vector quasi-equilibrium problem. However, the assumptions and proof methods of Theorem 3.1 in [11] are new and different from that in Theorem 3.1.

By the following example, we show that in the special case as Remark 3.2, all the assumptions of Theorem 3.1 are satisfied. But, Theorem 3.1 in [11] we cannot apply these conditions. It is because f is not $(-C)$-continuous.

Example 3.3 Let $X = Y = Z = \mathrm{R}$, $A = B = [0,2]$, $C = \mathrm{R}_+$

and let

$$K : A \times B \rightrightarrows A,\ T : A \times B \rightrightarrows B$$

and

$$f : A \times B \times A \to Z,\ g : B \times A \times B \to Z$$

be defined by

$$K(x,y) = T(x,y) = [0,2],$$

$$f(x,y,x_0) = g(y,x,y_0) = \begin{cases} [0,2], \text{ if } x_0 = y_0 = \frac{3}{2}, \\ [3,4] \text{ otherwise.} \end{cases}$$

We show that all assumptions of Theorem 3.1 are satisfied. However, f is not $(-C)$-continuous at $x_0 = y_0 = \frac{3}{2}$.

Thus, it gives case where Theorem 3.1 can be applied but Theorem 3.1 in [11] does not work.

Remark 3.4 If we let

$$f(x,y,x^*) = f(x^*,y) - f(x,y),$$
$$g(y,x,y^*) = g(x,y^*) - g(x,y)$$

with $x \in A,\ y \in A,\ x^* \in A,\ y^* \in B$ and replace C by Z-int C. Then, (SQVEP) will be changed to symmetric vector quasi-equilibrium problem which is studied in [2]. In [2] Fu obtained an existence result for symmetric vector quasi-equilibrium problem. However, the assumptions and proof methods of Theorem in [2] are also new and different from that in Theorem 3.1.

In the following example, we show that in the special case as Remark 3.4, all the assumptions of Theorem 3.1 are satisfied. But, Theorem 3.1 in [11] and Theorem in [2] can not be applied.

Example 3.5 Let $X = Y = Z = \mathrm{R}$, $A = B = [0,3], C = \mathrm{R}_+$ and let

$$K : A \times B \rightrightarrows A,\ T : A \times B \rightrightarrows B$$

and

$$f : A \times B \times A \to Z,\ g : B \times A \times B \to Z$$

be defined by

$$K(x,y) = T(x,y) = [0,3],$$
$$f(x,y,x_0) = g(y,x,y_0) = \begin{cases} [0,2], \text{ if } x_0 = y_0 = \frac{1}{4}, \\ [0,\frac{1}{3}] \text{ otherwise.} \end{cases}$$

We show that all assumptions of Theorem 3.1 are satisfied. However, f is neither C-continuous nor properly C-quasi-convex at $x_0 = y_0 = 1/4$. Therefore, Theorem 3.1 can be applied but Theorem 3.1 in [11] and Theorem in [2] do not work.

Theorem 3.6 *Suppose that for the problem* (SQVEP) *assumptions* (i), (ii), (iii) *and* (iv) *as in Theorem 3.1 and the condition* (v) *can be replaced by the following condition:*

(v') f and g are continuous.

Then, the (SQVEP) has a solution, i.e., there exists $(\bar{x},\bar{y}) \in A \times B$ *such that*

$$\bar{x} \in K(\bar{x},\bar{y}), \bar{y} \in T(\bar{x},\bar{y})$$

and

$$f(\bar{x},\bar{y},x^*) \in C, \forall x^* \in K(\bar{x},\bar{y}),$$
$$g(\bar{y},\bar{x},x^*) \in C, \forall y^* \in T(\bar{x},\bar{y}).$$

Moreover, the solution set of the (SQVEP) is closed.

Proof. We omit the proof since the technique is similar as that for Theorem 3.1 with suitable modifications.

CONCLUSION

The results presented in this paper improve and extend the main results in the literature. Some examples are given to illustrate our results. Hence our results, Theorem 3.1 and Theorem 3.6 have significant improvements.

ACKNOWLEDGMENTS

This research is funded by Ho Chi Minh City University of Technology-VNU-HCM under grant **number T-KHUD-2018-84**.

CONFLICT OF INTEREST

We declare that there is no conflict of whatsoever involved in publishing this research.

AUTHOR CONTRIBUTIONS

All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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