About the class of approximation by generalized characters

Dang Van Vinh[*]

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ABSTRACT

The common concept of approximation of algebraic system was given in the research of Russian academician Mal'cev A.I. In this article, Mal'cev showed a connection between the finitely approximation of algebraic system with respect to a given predicate and the problem of solvability of this predicate in the system. The notion of a finitely approximable semigroup is also mentioned. Some results about the approximation of semigroup were proposed by A. I. Mal'cev in the article. The research interest in an approximation of algebraic structures is inspired by Professor M.M. Lesokhin's work. There exist three main problems of approximation of algebraic structures. The first is finding conditions for the approximation of a structure A to a given structure B with respect to predicates using special kind of mappings from A to B, the second is finding a minimal approximated structure for a certain set of algebraic structures and the last is finding a class of approximated algebraic structures for a specified set of structures. Our research is to further investigate the problem of finding a class of approximation for a certain class of semigroups with respect to various predicates. In this research, we pay our attention to an important type of mappings that are generalized characters. Given a certain class K_1 of algebraic structures. We study a problem of finding a class K_2 of algebraic structures such that the class K_1 is approximable into K_2 with respect to various predicates by generalized characters from K_1 to K_2. The problem of minimization of approximation is also considered. Some theorems related to the problem of constructing an approximation class are obtained. The problem in question is much more complicated and actual than the approximation problem we have been studying before [1-5]. The results of the description of the approximation class play an important role in studying the solvability problem of the predicate P in the class of semigroups K. In particular, if the approximation class consists of finite semigroups, then this problem is solved positively. Even more difficult is the problem of determining the necessary conditions that class K_1 is an approximation class for a given class K with respect to different predicates.

Key words: Approximation, class of approximation, minimal approximation, generalized characters

Ho Chi Minh City University of Technology, VNU-HCM, Vietnam

Correspondence

Dang Van Vinh, Ho Chi Minh City University of Technology, VNU-HCM, Vietnam

Email: dangvvinh@hcmut.edu.vn

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INTRODUCTION

A problem of approximation of algebraic structures with respect to various predicates has been proposed and studied by Professor M. M. Lesokhin [6,7] and his students [1-14]. They have found a necessary and sufficient conditions for approximation of algebraic structures with respect to many important predicates, such as "equality of two elements", "belonging of an element in a substructure", "belonging of an element in the group", "the relation of regular conjugation", "Green $\ell-$, $\mathfrak{R}-$, $h-$, $d-$ equivalency", "belonging of an element in the monogenic subsemigroup", etc.

The problem of approximation of algebraic structures consists of three components. The first component is a set of algebraic structures such as groups, finite groups, semigroups, compact semigroups, fields,...; the second component is a set of important predicates; and the last component is a set of functions such as homomorphisms, continuous characters, continuous bicharacters, generalized characters, etc.

A formulation of a research of approximation is followed in this manner: we start with a structure B and consider a set Φ of all mappings from a structure A to the structure B. Then we select a predicate P and try to find out necessary conditions for approximation of A with respect to the predicate P by mapping from Φ. After that, we determine sufficient conditions for the approximation. Finally, we minimize the structure B so that A is still approximable into a proper substructure of the structure B.

In this research, we begin with an important class of structures K and a predicate P. Then we look for a class of structures K_1, such that K is approximable by generalized character from K into K_1 with respect to the given predicate P. This problem is much more complicated and has an important role for the study of the class of structures K. A particular difficulty stands out when we try to discover necessary conditions for the class K_1 to be a class of approximation for the given class K.

PRELIMINARIES

Definition 2.1. Let A be a semigroup and B be a commutative semigroup. A generalized character is a homomorphism from A into the semigroup B.

Let A and B be two algebraic structures with the same type, Φ be the set of all homomorphisms from A to B, and P be a predicate defined on the set that consists of A, all subsets $\delta(A)$ of A and all images of A and $\delta(A)$ under the characters from Φ.

Definition 2.2. [2] An algebraic structure A is said to be approximable by homomorphisms from Φ with respect to P, if for a pair of subsets A_1, A_2 from A such that $P(A_1, A_2)$ is false, there exists $\varphi \in \Phi$ such that $P(\varphi(A_1), \varphi(A_2))$ is also false.

Definition 2.3. [1,15] Let K_1 and K_2 be two classes of algebraic structures with the same type and P be a predicate.

The class K_2 is said to be a class of approximation with respect to the predicate P for the class K_1, if every algebraic structure from K_1 is approximable into K_2 with respect to P.

Definition 2.4. [1,15] If every structure from the class K_1 can be embedded into some structure from a class K_2, then we write $K_1 \prec K_2$

It's obvious that, if K_1 is a class of approximation for the class K and $K_1 \prec K_2$, then K_2 is also a class of approximation for the class K.

Definition 2.5. [1,15] Let S and K be two classes of algebraic structures with the same type and P be a predicate.

The class K is called a minimal class of approximation for the class S with respect to P, if the following two conditions hold:

(i) The class K is a class of approximation for the class S;

(ii) if a class K_1 is a class of approximation for S and $K_1 \prec K_2$, then $K \subset K_1$.

Definition 2.6. [2,5] Let Q be the set of all prime numbers. Let G_p, $p \in Q$ be the quasi-cyclic group of the type p^∞ with an identity e_p and with an operation denoted by $\otimes_p$.

Let us denote $C^* = \underset{p \in Q}{U} G_p$. Define in C^* a multiplication as follows.

$\forall a_p, a_q \in C^*$

$$a_p * a_q = \begin{cases} a_p \otimes_p a_q, \ if \ p = q; \\ a_{max\{p,q\}}, \ if \ p \neq q \ and \ max\{p,q\} > 3 \\ e_5, \ if \ p \neq q \ and \ max\{p,q\} = 3 \end{cases}$$

Direct calculation shows that $C^* = (C^*, *)$ is a semigroup.

Using G_p, $p \in Q$, we can create an infinite number of semigroups with different strutures. Let us denote

C by the set of all these semigroups. Interesting that, every semigroup from C is a minimal semigroup approximation for many classes with respect to common predicates.

Two types of semilattice: The first picture in Figure 1 illustrates the semigroup C^*. The second picture representes another semigroup formed from G_p, $p \in Q$.
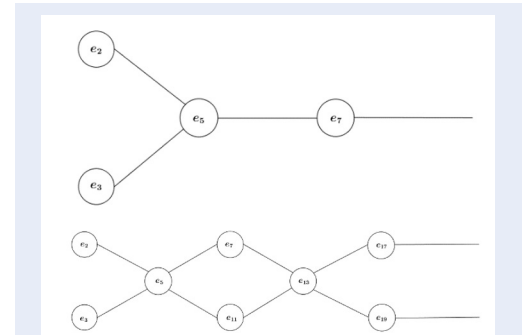

Figure 1: Two types of the semigroup C*

Definition 2.7. Let S be a semigroup and a, b be two elements of the semigroup S.

We define:

1/ alb, if $a \cup Sa = b \cup Sb$. In other words, $S^1a = S^1b$.

2/ $a \mathfrak{R} b$, if $a \cup aS = b \cup bS$ or $aS^1 = bS^1$.

3/ The relation l and $\mathfrak{R}$ are commute and the relation $l \circ \mathfrak{R} = \mathfrak{R} \circ l$ is denoted by d relation.

4/ h-relation is $h = l \cap \mathfrak{R}$ (Clifford *et al.* (1977), pages 47 and 48) [16].

CLASS OF APPROXIMATION

Vinh *et al.* (2019) have proved that the necessary and sufficient condition for the class S_1 to be a class of approximation with respect to the predicate of equality of two elements for the class C_1 of commutative semigroups of idempotents is that the class S_1 must contain a semigroup that is isomorphic to the semigroup of two elements $B = \{0; 1\}$. Vinh *et al.* (2019) have found a necesary and sufficient condition for the class S_2 to be a class of approximation with respect to the predicate of the possible belonging of an element to a subsemigroup for the class C_1 is that the class S_2 contains a semigroup that is isomorphic to the semilattice of three elements $T = \{e_1, e_2, e_3\}$, where $e_3 = e_3 e_1 = e_3 e_2, e_2 = e_2 e_1$.

In this paper, we continue our research about the class of approximation for other classes of semigroups with respect to different predicates.

Theorem 3.1. Let X_1 be the class of commutative and separative semigroups and Y_1 be a class of semigroups. If some semigroup from the class Y_1 is isomorphic to

the semigroup C^*, then the class Y_1 is a class of approximation for the class X_1 with respect to the predicate of equality of two elements.

Proof. Assume that Y_1 contains a semigroup M that is isomorphic to the semigroup C^*.

Let A be a semigroup from the class X_1.

Because A is commutative, (see the book of Clifford *et al.* (1977), Theorem 4.13, page 132), $A = \underset{e \in E}{U} A_e$ is the expression of A as a semilattice of its archimedean components A_e. The semigroup A is separative, see the book of Clifford *et al.* (1977), Theorem 4.13, page 134, A can be embedded in a semigroup $T = UG_e$, where G_e is a group that contains A_e.

Assume that $a, b \in A$ and $a \neq b$. Suppose that $a \in A_e$, $b \in A_b$. There are two possible cases.

Case 1: $e_a \neq e_b$. Then $e_a e_b \neq e_a$ or $e_a e_b \neq e_b$. Without loss of generality, assume that $e_a e_b \neq e_a$.

We create a mapping $\psi : A \to C^*$ in the following way:

$$\forall x \in A, \psi(x) = \begin{cases} a_{q_0} \ if \ x \in A_{e_x} \ and \ e_x e_a = e_a; \\ a_{p_0} \ if \ x \in A_{e_x} \ and \ e_x e_a \neq e_a \end{cases}$$

where p_0, q_0 are two fixed prime numbers such that $p_0 > q_0 \geq 3$.

The mapping ψ is a homomorphism.

Indeed, for any $x \in A_{e_x}$ and $y \in A_{e_y}$

a/ If $e_x e_a = e_a$ and $e_y e_a = e_a$, then $e_{xy} e_a = e_x e_y e_a = e_x e_a = e_a$. It follows that $\psi(xy) = e_{q_0} = e_{q_0} e_{q_0} = \psi(x) \psi(y)$

b/ If $e_x e_a \neq e_a$ and $e_y e_a = e_a$, then $e_{xy} e_a = e_x e_y e_a = e_x e_a \neq e_a$. Hence $\psi(xy) = e_{p_0} = e_{p_0} e_{q_0} = \psi(x) \psi(y)$.

c/ If $e_x e_a \neq e_a$ and $e_y e_a \neq e_a$, then $e_{xy} e_a \neq e_a$(if $e_a = e_{xy} e_a = e_x e_y e_a = e_x e_x e_y e_a = e_x e_{xy} e_a = e_x e_a \neq e_a$. It is impossible)

We obtain $\psi(xy) = e_{p_0} = e_{p_0} e_{p_0} = \psi(x) \psi(y)$. Thus ψ is a generalized character from A to C^*, and $\psi(a) = e_{q_0} \neq e_{p_0} = \psi(b)$.

Case 2. The two elements *a, b* belong to the same archimedean component A_{e_0} of the semigroup A.

Because $a \neq b$, $g = ab^{-1} \neq e_0$, where $g \in G_{e_0}$ and G_{e_0} is a subgroup that contains the semigroup A_{e_0}.

Assume that h is a height of the element g. By the book of Lazlo Fuchs (1967), Lemma 26.4, page 85, there exists a maximum subgroup H of the group G_{e_0}, such that $G_{e_0}/H \approx C\left(q_0^{h+1}\right)$ and $g \notin H$.

It follows that there is a homomorphism $\varphi : G_{e_0} \to C\left(q_0^{h+1}\right)$ satisfying $\varphi(H) = e_{q_0}$ and $\varphi(g) \neq e_{q_0}$.

Because $g = ab^{-1}, \varphi\left(ab^{-1}\right) \neq e_{q_0} \leftrightarrow \varphi(a) \varphi\left(b^{-1}\right) \neq e_{q_0}$.

Finally, $\varphi(a) \neq \varphi(b)$.

Note that $C\left(q_0^{h+1}\right)$ is a subgroup of the group $C(q_0)$ and $C(q_0) \approx G_{q_0}$, where G_{q_0} is a subgroup of the semigroup C^*.

From this point, we have a generalized character $\varphi : G_{e_0} \to C^*$ and $e_0 = e_{q_0}$.

Next, we create a mapping $\psi : T \to C^*$

$$\forall x \in T, \psi(x) = \begin{cases} e_{p_0} \ if \ x \in G_{e_x} \ and \ e_x e_0 = e_0 \\ \psi(x e_0) \ if \ x \in G_{e_x} \ and \ e_x e_0 \neq e_0 \end{cases}$$

where p_0 is a fixed prime number, such that $p_0 \neq 3$ and $p_0 > q_0$.

If $x \in G_{e_x}$ and $e_x e_0 = e_0$, then $x e_0 \in G_{e_0}$.

Direct calculation shows that ψ is a generalized character from the semigroup T into the semigroup C^* satisfying the condition $\psi(a) = \varphi(a) \neq \varphi(b) = \psi(b)$.

It is obvious that, the mapping ψ is a generalized character from A into C^* and this homomorphism separates the two different elements of the semigroup A.

Theorem 3.2. Let X_2 be the class of semigroups A such that A can be embedded into a semilattice of left simple semigroups and Y_2 be a class of semigroups. If some semigroup from the class Y_2 contains a semigroup that is isomorphic to the semigroup C^*, then the class Y_2 is a class of approximation for the class X_2 with respect to the Green relation l-equivalency.

Proof. Assume that Y_2 contains a semigroup M that is isomorphic to the semigroup C^*.

Let A be a semigroup of the class X_2. The semigroup A can be embedded into a semilattice of left simple semigroups.

Let *a, b* be two elements of A such that $a \bar{l} b$.

Then $b \notin L_a$, where L_a is L class, that contains the element a.

Consider a set $J_a = \left\{x \in A | a x \bar{l} a\right\}$.

Let $c \in J_a$, $d \in A$. Assume that $cd \notin J_a$. It is equivalent that $acdla$.

Because $cdldc$, $L_a L_{cd} = L_a L_{dc}$, hence $acd = adc$.

From $alacd$ and $acdladc$, we have $aladc$. It implies that $acl(adc)c$.

In the other hand, $(adc)c = (ad)(cc) \in L_{ad}L_{cc} \subset L_{ad}L_c \subseteq L_{adc}$.

And we have got $acladc$.

Since $adcla$, $acla$ and $c \notin J_a$. It contradicts the assumption.

Thus $cd \in J_a$ and it shows that J_a is an ideal.

Suppose that $c, d \in A \backslash J_a$. Hence $ac \in L_a$ and $ad \in L_a$. Then $acd \in L_a$. It follows that $cd \notin J_a$.

Thus $A \backslash J_a$ is a subsemigroup and J_a is completely isolated ideal.

Suppose that $a, b \in A$, such that $b \notin L_a$.

Let us consider a mapping $\varphi : A \to C^*$ that is defined in the following way:

$$\forall x \in A, \varphi(x) = \begin{cases} e_q, if \ x \in J_a; \\ e_p, if \ x \notin J_a \end{cases}$$

where p and q are two fixed prime numbers such that $q > p \geq 3$.

Due to J_a is a completely isolated ideal, φ is a generalized character. Moreover $\varphi(a) = e_p$ (because $aa \in L_a, a \notin J_a$ and $\varphi(b) = e_q$).

Because $b \notin L_a$, $ab \notin L_a$.

It is obvious that $e_p \bar{l} e_q$ in C^*.

At the end, we obtain $\varphi(a)\bar{l}\varphi(b)$. It means that the semigroup A is approximable with respect to the Green relation l-equivalency.

From the proof of the Theorem 3.2, we have a following result:

Let $C^{**} = \{e_p; e_q\}$, where $p > q \geq 3$ be a subsemigroup of the semigroup C^*.

Theorem 3.3. A class of semigroups Y_3 is a class of approximation for the class X_2 with respect to the Green relation l-equivalency if and only if some of semigroup from the class Y_3 is isomorphic to the semigroup C^{**}.

Within the framework of this paper, I only present the results with proof for several number of predicates. Apart from the results for L-equivalences above, I also have found other results with respect to other Green's relations: R-equivalency, D-equivalence and H- equivalence.

Theorem 3.4. Let C_3 be the class of semigroups A such that A can be embedded into a semilattice of right simple semigroups. If some of semigroup from a class S_3 contains a semigroup that is isomorphic to the semigroup C^*, then the class S_3 is a class of approximation for the class S_3 with respect to the Green relation $\mathfrak{R}$-equivalency.

CONCLUSION

One of the important directions in modern algebra is the study of not only the algebraic system itself, but also the systems derived from it. Using the relationships established in the paper, it is possible to extend the obtained results to other classes of algebraic structures, other sets of homomorphisms and other predicates.

Finding a minimal class of approximation for a well-known class of structures with respect to important predicates is extremely sophisticated. Vinh *et al.* (2018) have proved that the semigroup C^* is a minimal semigroup of approximation for the class C_4 of commutative, regular and periodic semigroups with respect to the predicate P of the possible of belonging an element to a subsemigroup and in the book of Korabelsh'chikova *et al.* (2013), the authors described another semigroup that played a role as a minimal semigroup of approximation for the class C_4 with respect to P. In order to find a class of approximation for the class C_4 we have to clarify common features between these two semigroups.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests regarding the publishing of this paper.

AUTHORS' CONTRIBUTIONS

Dang Van Vinh is in charge of all research content presented in the article.

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